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A 2-categorial generalization of the concept of institution

Abstract. After defining, for each many-sorted signature $\mathbf{\Sigma} = (S, \Sigma)$, the category $\mathbf{Ter}(\mathbf{\Sigma})$, of generalized terms for $\mathbf{\Sigma}$ (which is the dual of the Kleisli category for $\mathbb{T}_{\mathbf{\Sigma}}$, the monad in $\mathbf{Set}^S$ determined by the adjunction $\mathbf{T}_{\mathbf{\Sigma}} \dashv \mathrm{G}_{\mathbf{\Sigma}}$ from $\mathbf{Set}^S$ to $\mathbf{Alg}(\mathbf{\Sigma})$, the category of $\mathbf{\Sigma}$-algebras), we assign, to a signature morphism $\mathbf{d}$ from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$, the functor $\mathbf{d}_\diamond$ from $\mathbf{Ter}(\mathbf{\Sigma})$ to $\mathbf{Ter}(\mathbf{\Lambda})$. Once defined the mappings that assign, respectively, to a many-sorted signature the corresponding category of generalized terms and to a signature morphism the functor between the associated categories of generalized terms, we state that both mappings are actually the components of a pseudo-functor Ter from $\mathbf{Sig}$ to the 2-category $\mathbf{Cat}$. Next we prove that there is a functor $\mathrm{Tr}^{\mathbf{\Sigma}}$, of realization of generalized terms as term operations, from $\mathbf{Alg}(\mathbf{\Sigma}) \times \mathbf{Ter}(\mathbf{\Sigma})$ to $\mathbf{Set}$, that simultaneously formalizes the procedure of realization of generalized terms and its naturalness (by taking into account the variation of the algebras through the homomorphisms between them). We remark that from this fact we will get the invariance of the relation of satisfaction under signature change. Moreover, we prove that, for each signature morphism $\mathbf{d}$ from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$, there exists a natural isomorphism $\theta^{\mathbf{d}}$ from the functor $\mathrm{Tr}^{\mathbf{\Lambda}} \circ (\mathrm{Id} \times \mathbf{d}_\diamond)$ to the functor $\mathrm{Tr}^{\mathbf{\Sigma}} \circ (\mathbf{d}^* \times \mathrm{Id})$, both from the category $\mathbf{Alg}(\mathbf{\Lambda}) \times \mathbf{Ter}(\mathbf{\Sigma})$ to the category $\mathbf{Set}$, where $\mathbf{d}^*$ is the value at $\mathbf{d}$ of the arrow mapping of a contravariant functor Alg from $\mathbf{Sig}$ to $\mathbf{Cat}$, that shows the invariant character of the procedure of realization of generalized terms under signature change. Finally, we construct the many-sorted term institution by combining adequately the above components (and, in a derived way, the many-sorted specification institution), but for a strict generalization of the standard notion of institution.

Keywords: Many-sorted algebra, generalized term, Kleisli construction, 2-institution on a category, institution on a category.

1. Introduction.

The theory of institutions of Goguen and Burstall, which arose within theoretical computer science, in response to the proliferation of logics in use there, is a categorial formalization of the *semantic* aspect of the intuitive notion of "logical system", and it has as objectives, according to Goguen and Burstall in [15]: "(1) To support as much computer science as possible independently of the underlying logical system, (2) to facilitate the transfer of results (and artifacts such as theorem provers) from one logical system to another, and (3) to permit combining a number of different logical systems".

We recall that Goguen and Burstall in [12], p. 229, define an *institution* as a category **Sign**, of signatures, a functor Sen from **Sign** to **Set**, giving the set of *sentences* over a given signature, a functor Mod from **Sign** to $\mathbf{Cat}^{\mathrm{op}}$, giving the category of *models* of a given signature, and, for each $\Sigma \in \mathbf{Sign}$, a satisfaction relation $\models_\Sigma \subseteq |\mathbf{Mod}(\Sigma)| \times \mathrm{Sen}(\Sigma)$, where $|\cdot|$ is the endofunctor of $\mathbf{Cat}$ which sends a category to the discrete category on its set of objects, such that, for each morphism $\varphi\colon \Sigma \longrightarrow \Sigma'$, the

Satisfaction Condition. $\mathbf{M}' \models_{\Sigma'} \varphi(e)$ iff $\varphi(\mathbf{M}') \models_\Sigma e$,

holds for each $\mathbf{M}' \in |\mathbf{Mod}(\Sigma')|$ and each $e \in \mathrm{Sen}(\Sigma)$ (Let us notice the abuse of notation on the part of Goguen and Burstall about "$\varphi(e)$" and "$\varphi(\mathbf{M}')$" in their formulation of the *Satisfaction Condition*. They should literally be "$\mathrm{Sen}(\varphi)(e)$" and "$\mathrm{Mod}(\varphi)(\mathbf{M}')$", respectively). Later on, in [15], p. 316, they define an *institution* as a category $\mathbf{Sign}$, of signatures, a functor Sen from $\mathbf{Sign}$ to $\mathbf{Cat}$ (observe the large-scale change from $\mathbf{Set}$ to $\mathbf{Cat}$ in this definition, we emphasize), giving *sentences* and *proofs* over a given signature, a functor Mod from $\mathbf{Sign}$ to $\mathbf{Cat}^{\mathrm{op}}$, giving the category of *models* of a given signature, and, for each $\Sigma \in |\mathbf{Sign}|$, a satisfaction relation $\models_\Sigma \subseteq |\mathbf{Mod}(\Sigma)| \times |\mathbf{Sen}(\Sigma)|$ such that

Satisfaction Condition: $\mathbf{M}' \models_{\Sigma'} \mathrm{Sen}(\varphi)s$ iff $\mathrm{Mod}(\varphi)\mathbf{M}' \models_\Sigma s$, for each $\varphi\colon \Sigma \longrightarrow \Sigma'$ in $\mathbf{Sign}$, $\mathbf{M}' \in |\mathbf{Mod}(\Sigma')|$ and $s \in |\mathbf{Sen}(\Sigma)|$, and

Soundness Condition: if $\mathbf{M} \models_\Sigma s$, then, for each $\mathbf{M} \in |\mathbf{Mod}(\Sigma)|$ and each $s \longrightarrow s' \in \mathbf{Sen}(\Sigma)$, $\mathbf{M} \models_\Sigma s'$.

Besides, the same authors, in [15], p. 327, define, for a category $\mathbf{V}$, a *generalized* $\mathbf{V}$*-institution* as a pair of functors Mod, from $\mathbf{Sign}^{\mathrm{op}}$ to $\mathbf{Cat}$, and Sen, from $\mathbf{Sign}$ to $\mathbf{Cat}$, with an extranatural transformation $\models$ from $|\mathrm{Mod}(\cdot)| \times \mathrm{Sen}(\cdot)$ to $\mathbf{V}$. Observe that the second concept of institution falls under this last one because, taking as $\mathbf{V}$ the category $\mathbf{2}$, with two objects and just one morphism not the identity, the existence of an extranatural transformation from $|\mathrm{Mod}(\cdot)| \times \mathrm{Sen}(\cdot)$ to $\mathbf{2}$ is equivalent to the above satisfaction and soundness conditions. For a recent and thorough treatment of the theory of institutions we refer the reader to the first part of [8].

This article deals with a category-theoretic investigation of various concepts around the many-sorted equational logic and its connection with the work of Goguen and Burstall on institutions. This has as a consequence that the notion of institution is generalized towards two directions: (1) by parameterizing its "truth-value structure" by an arbitrary category, and (2) by allowing a 2-category structure on signatures, reflected in an appropriate way on the mappings Mod and Sen. The first direction of generalization integrates terms as "sentences" that are more basic and exhibit more structure than actual sentences, and the second allows for a very flexible notion of specification morphism and of equivalence between specifications.

Next we sketch an example (accurately developed in the second section) that suggests the need to generalize the notion of institution. For **Sig**, the category of many-sorted signatures, and **Cat**, the category of $\boldsymbol{\mathcal{U}}$-categories, for a fixed Grothendieck universe $\boldsymbol{\mathcal{U}}$, the many-sorted term institution, $\mathfrak{Tm}$, consists of: (1) the category **Sig**, (2) the contravariant functor Alg from **Sig** to **Cat**, which sends $\mathbf{\Sigma}$ in **Sig** to $\mathbf{Alg}(\mathbf{\Sigma})$, the category of $\mathbf{\Sigma}$-algebras, (3) the pseudo-functor Ter from **Sig** to **Cat**, which sends $\mathbf{\Sigma}$ in **Sig** to $\mathbf{Ter}(\mathbf{\Sigma})$, the category of generalized terms for $\mathbf{\Sigma}$, and (4) the pair (Tr, θ), where Tr is the family $(\mathrm{Tr}^{\mathbf{\Sigma}})_{\mathbf{\Sigma}\in\mathbf{Sig}}$, where, for each $\mathbf{\Sigma}$ in **Sig**, $\mathrm{Tr}^{\mathbf{\Sigma}}$ is the functor from $\mathbf{Alg}(\mathbf{\Sigma}) \times \mathbf{Ter}(\mathbf{\Sigma})$ to **Set** that formalizes the realization of terms as term operations on algebras, and θ the family $(\theta^{\mathbf{d}})_{\mathbf{d}\in\mathrm{Mor}(\mathbf{Sig})}$, where, for each morphism $\mathbf{d}$ from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$ in **Sig**, $\theta^{\mathbf{d}}$ is a natural isomorphism between two suitable functors from $\mathbf{Alg}(\mathbf{\Lambda}) \times \mathbf{Ter}(\mathbf{\Sigma})$ to **Set**, which shows the invariant character of the procedure of realization of terms under signature change.

This, together with several similar examples, has led us to consider the following generalization of the concept of institution, parameterized by a given category $\mathbf{C}$: a quadruple $(\mathbf{Sig}, \mathrm{Mod}, \mathrm{Sen}, (\alpha, \beta))$, where **Sig** is a category, Mod a pseudo-functor from $\mathbf{Sig}^{\mathrm{op}}$ to **Cat**, Sen a pseudo-functor from **Sig** to **Cat**, α consists, for each object $\mathbf{\Sigma} \in \mathbf{Sig}$, of a functor $\alpha_{\mathbf{\Sigma}}$ from $\mathbf{Mod}(\mathbf{\Sigma}) \times \mathbf{Sen}(\mathbf{\Sigma})$ to $\mathbf{C}$, β consists, for each morphism $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ in **Sig**, of a natural isomorphism $\beta_{\mathbf{d}}$ from the functor $\alpha_{\mathbf{\Lambda}} \circ (\mathrm{Id}_{\mathbf{Mod}(\mathbf{\Lambda})} \times \mathrm{Sen}(\mathbf{d}))$ to the functor $\alpha_{\mathbf{\Sigma}} \circ (\mathrm{Mod}(\mathbf{d}) \times \mathrm{Id}_{\mathbf{Sen}(\mathbf{\Sigma})})$, where α and β are subject to satisfy some expected naturalness conditions. Obviously, any ordinary institution $(\mathbf{Sig}, \mathrm{Mod}, \mathrm{Sen}, \models)$ is an institution. The richer structure of institutions accommodate "term-institutions" such as the motivating example (and also seem to accommodate non-classical logics) integrating terms as "sentence" that are more basic and exhibit more structure than actual sentences built of terms. In addition we have defined a notion of 2-institution, also parameterized by a given category $\mathbf{C}$, which is roughly obtained by allowing the category **Sig** to be a 2-category, letting Mod and Ter be pseudo-functors, and (α, β) a pseudo-extranatural transformation which allows for a very flexible notion of specification morphism and equivalence. Now, a 2-category structure on the class of many-sorted signatures is not easy to motivate, namely it is not clear what the 2-cells should stand for. However, in [6], we have appropriately motivated this new concept using as 1-cells the polyderivors, which generalize (1) the standard morphisms, (2) the derivors between many-sorted signatures, and (3) the families of basic mapping-formulas defined by Fujiwara in [10] for the single-sorted case, and as 2-cells the transformations between polyderivors, which generalize the equivalences between families of basic mapping-formulas also defined by Fujiwara in [11] for the single-sorted case.

Although not covered in this work, we notice that many-sorted signatures, polyderivors, and transformations between polyderivors yield a 2-category $\mathbf{Sig}_{\mathfrak{pd}}$ which is the foundation for a very important example of 2-institution on **Set**, precisely $\mathfrak{Tm}_{\mathfrak{pd}}$, the term 2-institution of Fujiwara. Moreover, from $\mathbf{Sig}_{\mathfrak{pd}}$ it is possible to obtain in a derived way a 2-category $\mathbf{Spf}_{\mathfrak{pd}}$, of many-sorted specifications, 1-cells those polyderivors between the underlying many-sorted signatures of the specifications that are compatible with the equations, and 2-cells from a 1-cell into another a convenient class of transformations between the polyderivors, which provides another example of 2-institution, exactly $\mathfrak{Spf}_{\mathfrak{pd}}$, the specification 2-institution of Fujiwara. All that is needed to prove the above assertions can be found in [6]. Let us point out that the importance of defining more and more general (yet sufficiently insightful) morphisms between specifications resides in the flexibility of algebraic specification notions such as embedding, refinement or parametricity. Flexible specification morphisms yield flexible notions of equivalence between specifications. In mathematics, equivalent classes of algebras need not have the same signature, but rather have, natural, back and forth mutual embeddings. In algebraic specifications, equivalence is sometimes allowed to be even more flexible, e.g., the behavioral equivalence. What makes two (many-sorted, first order) specifications equivalent? An apparent general answer would be the existence of an isomorphism between their classes of models (or of an isomorphism between behavioral classes). However, this answer is seen to be extremely inadequate as soon as one notices that any two proper classes are isomorphic. The notion of equivalence has to have the models as final target, but it needs to be anchored into syntax to make the equivalence constructible and usable. The 2-category structure given by the polyderivors represents progress towards such general and insightful specification equivalence.

We emphasize that attempts (1) to formalize the insight that syntax of logics can be described as free algebras for a suitable signature and to define logical systems based on such a formalization, or (2) to revise the concept of institution in such a way that the inner structure of formulas and, especially, generalized terms (or substitutions) are not entirely new as shown by the following examples. In addition to charters and parchments as in [15], galleries, see [25] (about which Goguen and Burstall in [15], p. 330, say: "Galleries and the extranatural transformation formulation of institutions, motivate our concept of generalized institution"), context institutions, see [27] (and [32]), and foundations, see [28]. However, none of them attempts a generalization of the notion of institution like that proposed in this article along the lines of 2-categories and associated concepts.

Next we proceed to succinctly describe the contents of the following sections.

The main goal of the *second section* is to construct the many-sorted term institution. To attain such a goal we begin by defining **MSet**, the category of many-sorted sets and many-sorted mappings, **Sig**, the category of standard many-sorted signatures, and **Alg**, the category of standard many-sorted algebras and morphisms between many-sorted algebras of *different* many-sorted signatures, through the Ehresmann-Grothendieck's construction applied, respectively, to suitable contravariant functors MSet, Sig, and Alg. Then we prove that **Alg** is concrete and uniquely transportable through a "forgetful" functor G into $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$, and that the functor G has a left adjoint $\mathbf{T}\colon \mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig} \longrightarrow \mathbf{Alg}$ which transforms objects of $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$ into labelled term algebras in **Alg** and morphisms of $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$ into translators between the associated labelled term algebras in **Alg**. On the basis of the functor **T** we define, for every many-sorted signature $\mathbf{\Sigma}$, the category $\mathbf{Ter}(\mathbf{\Sigma})$, of generalized terms for $\mathbf{\Sigma}$, as the dual of the Kleisli category for $\mathbb{T}_{\mathbf{\Sigma}}$ (the standard monad derived from the adjunction between the category $\mathbf{Alg}(\mathbf{\Sigma})$, of $\mathbf{\Sigma}$-algebras, and the category $\mathbf{Set}^{S}$, of S-sorted set), and we extend this procedure to a *pseudo-functor* Ter from **Sig** to **Cat** which formalizes the procedure of translation for many-sorted terms. Then, to account exactly for the invariant character of the procedure of realization of the many-sorted terms in the many-sorted algebras, under change of many-sorted signature, we show that there exists a *pseudo-extranatural* transformation from a *pseudo-functor* obtained from Alg and Ter to the functor $\mathrm{K}_{\mathbf{Set}}$, which picks **Set**, both defined on $\mathbf{Sig}^{\mathrm{op}} \times \mathbf{Sig}$ and taking values in the 2-*category* **Cat**. Finally, after generalizing the concept of institution by means, essentially, of the notion of *pseudo-extranatural* transformation from a *pseudo-functor* to a constant functor, we obtain $\mathfrak{Tm}$, the many-sorted term institution on **Set**.

In the *third section* we begin by defining, for a many-sorted signature $\mathbf{\Sigma}$, the concept of $\mathbf{\Sigma}$-equation, but for the generalized terms in the category $\mathbf{Ter}(\mathbf{\Sigma})$, the relation of satisfaction between many-sorted algebras and $\mathbf{\Sigma}$-equations, the consequence operator $\mathrm{Cn}_{\mathbf{\Sigma}}$, and by translating, for a morphism between many-sorted signatures, equations for the source many-sorted signature into equations for the target many-sorted signature. Then we continue with the proof of the satisfaction condition and, after defining a convenient pseudo-functor from **Sig** to $\mathbf{Cat}_{\boldsymbol{\mathcal{V}}}$, for an adequate Grothendieck universe $\boldsymbol{\mathcal{V}}$, we obtain $\mathfrak{LEq}$, the many-sorted equational institution on **2**.

Following this, after defining the category $\mathbf{Spf}$, of many-sorted specifications and many-sorted specification morphisms, we prove the existence of a contravariant functor, $\mathrm{Alg}^{\mathrm{sp}}$, and of a *pseudo-functor*, $\mathrm{Ter}^{\mathrm{sp}}$, from $\mathbf{Spf}$ to $\mathbf{Cat}$, that extend Alg and Ter, respectively. Then we state that from $\mathbf{Spf}^{\mathrm{op}} \times \mathbf{Spf}$ to the 2-category $\mathbf{Cat}$ there exists a *pseudo-functor*, obtained from $\mathrm{Alg}^{\mathrm{sp}}$ and $\mathrm{Ter}^{\mathrm{sp}}$, and a *pseudo-extranatural* transformation from it to the functor $\mathrm{K}_{\mathbf{Set}}$, and from this we obtain $\mathfrak{Spf}$, the many-sorted specification institution on $\mathbf{Set}$, and an institution morphism from $\mathfrak{Spf}$ to $\mathfrak{Tm}$ (actually, $\mathfrak{Tm}$ is a retract of $\mathfrak{Spf}$).

Every set we consider, unless otherwise stated, will be a $\boldsymbol{\mathcal{U}}$-small set or a $\boldsymbol{\mathcal{U}}$-large set, i.e., an element or a subset, respectively, of a Grothendieck universe $\boldsymbol{\mathcal{U}}$ (as defined, e.g., in [23], p. 22), fixed once and for all. Besides, we agree that $\mathbf{Set}$ denotes the category which has as set of objects $\boldsymbol{\mathcal{U}}$ and as set of morphisms the subset of $\boldsymbol{\mathcal{U}}$ of all mappings between $\boldsymbol{\mathcal{U}}$-small sets, and, depending on the context, that $\mathbf{Cat}$ denotes either, the category of the $\boldsymbol{\mathcal{U}}$-categories (i.e., categories $\mathbf{C}$ such that the set of objects of $\mathbf{C}$ is a subset of $\boldsymbol{\mathcal{U}}$, and the hom-sets of $\mathbf{C}$ elements of $\boldsymbol{\mathcal{U}}$), and functors between $\boldsymbol{\mathcal{U}}$-categories, or the 2-category of the $\boldsymbol{\mathcal{U}}$-categories, functors between $\boldsymbol{\mathcal{U}}$-categories, and natural transformations between functors.

On the other hand, in all that follows we use standard concepts and constructions from category theory, see e.g., [1], [4], [9], [18], [21], and [23]; classical universal algebra, see e.g., [7], [17], and [20]; categorical universal algebra, see e.g., [2] and [22]; many-sorted algebra, see e.g., [2], [3], [16], [19], and [24]; and "Goguen school" of institutions, see e.g., [13], [14], [8], [26], and [30]. Nevertheless, we have generically adopted the following notational and terminological conventions. For a set B, a family of sets $(A_i)_{i\in I}$, and a family of mappings $(f_i)_{i\in I}$ in $\prod_{i\in I}\mathrm{Hom}(B, A_i)$, we denote by $\langle f_i\rangle_{i\in I}$ the unique mapping from B to $\prod_{i\in I} A_i$ such that, for every $i \in I$, $f_i = \mathrm{pr}_i \circ \langle f_i\rangle_{i\in I}$, where pr_i is the canonical projection from $\prod_{i\in I} A_i$ to A_i. For a set S we agree upon denoting by $\mathbf{S}^\star = (S^\star, \curlywedge, \lambda)$ the *free monoid on* S, where $S^\star$, the underlying set of $\mathbf{S}^\star$, is $\bigcup_{n\in\mathbb{N}} S^n$, the set of all *words on* S, $\curlywedge$ the *concatenation* of words on S, and λ the *empty word* on S. For a word w on S, $|w|$ is the length of w. More specific notational conventions will be included and explained in the successive sections.

2. The many-sorted term institution.

Our main aim in this section is to show that the concept of "derived operation of an algebra", also known as "term operation of an algebra", elemental as it is, but fundamental for universal algebra, can be naturally subsumed under the notion of institution (see for this notion, e.g., [15]), provided that an institution is meant not to be an extranatural transformation (as in [15]) but a pseudo-extranatural transformation (as defined at the end of this section).

To attain the aim just mentioned we begin by a careful examination of the different types of things that are involved around it, namely many-sorted sets, signatures, algebras, terms, and generalized institutions. More specifically, in this section we define the category **MSet** of many-sorted sets, in which the many-sorted sets will be labelled with the sets of sorts, by applying the Ehresmann-Grothendieck's construction (henceforth abbreviated to EG-construction) (see [9], pp. 89–91 and [18], pp. (sub.) 175–177) to a contravariant functor MSet from **Set** to **Cat**. Following this we define the categories **Sig**, of many-sorted signatures, and **Alg**, of many-sorted algebras, by applying also the EG-construction to suitable contravariant functors Sig from **Set** to **Cat**, and Alg from **Sig** to **Cat**, respectively.

Besides we prove the existence of a left adjoint **T** to a "forgetful" functor G from **Alg** to **MSet** $\times_{\mathbf{Set}}$ **Sig**, and from this left adjoint **T** we define a pseudo-functor Ter from **Sig** to **Cat** which formalizes the procedure of translation for many-sorted terms.

Finally, to account exactly for the invariant character of the realization of many-sorted terms in many-sorted algebras under change of many-sorted signature, we prove the existence of a pseudo-extranatural transformation from a pseudo-functor on $\mathbf{Sig}^{\mathrm{op}} \times \mathbf{Sig}$ to **Cat**, induced by Alg and Ter, to the functor $\mathrm{K}_{\mathbf{Set}}$, between the same categories, which picks **Set**. Then, after providing a generalization of the ordinary concept of institution, we prove that the pseudo-extranatural transformation is, to be more precise, part of an institution on **Set**, the so-called many-sorted term institution.

Before stating the first proposition of this section, we agree upon calling, henceforth, for a set (of sorts) $S \in \boldsymbol{\mathcal{U}}$ (recall that $\boldsymbol{\mathcal{U}}$ is the set of objects of **Set**), the objects of the category $\mathbf{Set}^S$ (i.e., the elements $A = (A_s)_{s\in S}$ of $\boldsymbol{\mathcal{U}}^S$) *$S$-sorted sets*; and the morphisms of the category $\mathbf{Set}^S$ from an S-sorted set A into another B (i.e., the ordered triples (A, f, B), abbreviated to $f\colon A \longrightarrow B$, where f is an element of $\prod_{s\in S} \mathrm{Hom}(A_s, B_s)$) *$S$-sorted mappings from A to B*. Furthermore, we also agree that a pseudo-functor F from a *category* $\mathbf{C}$ to a 2-*category* $\mathbf{D}$ consists of the following data:

1. An object mapping $F\colon \mathrm{Ob}(\mathbf{C}) \longrightarrow \mathrm{Ob}(\mathbf{D})$.
2. For every $x, y \in \mathbf{C}$, an hom-mapping F from the set of morphisms $\mathrm{Hom}_{\mathbf{C}}(x, y)$ to the set of morphisms $\mathrm{Hom}_{\mathbf{D}}(F(x), F(y))$.
3. For every morphisms $f\colon x \longrightarrow y$ and $g\colon y \longrightarrow z$ in $\mathbf{C}$, an isomorphic 2-cell $\gamma^{f,g}$ from $F(g) \circ F(f)$ to $F(g \circ f)$.

4. For every $x \in \mathbf{C}$, an isomorphic 2-cell ν^x from $\mathrm{id}_{F(x)}$ to $F(\mathrm{id}_x)$.

These data must satisfy the following coherence axioms:

1. For morphisms $f\colon x \longrightarrow y$, $g\colon y \longrightarrow z$, and $h\colon z \longrightarrow t$ in $\mathbf{C}$,

$$\gamma^{g\circ f,h} \circ (\mathrm{id}_{F(h)} * \gamma^{f,g}) = \gamma^{f,h\circ g} \circ (\gamma^{g,h} * \mathrm{id}_{F(f)}).$$

2. For a morphism $f\colon x \longrightarrow y$ in $\mathbf{C}$,

$$\mathrm{id}_{F(f)} = \gamma^{\mathrm{id}_x,f} \circ (\mathrm{id}_{F(f)} * \nu^x) \text{ and } \mathrm{id}_{F(f)} = \gamma^{f,\mathrm{id}_y} \circ (\nu^y * \mathrm{id}_{F(f)}).$$

In the following proposition, that is basic for a great deal of what follows, for a mapping φ from S to T, we prove the existence of an adjunction $\coprod_\varphi \dashv \Delta_\varphi$ from the category of S-sorted sets to the category of T-sorted sets, as well as the existence of a contravariant functor MSet and of a pseudo-functor $\mathrm{MSet}^{\amalg}$ (related, respectively, to the right and left components of the adjunction) from **Set** to **Cat**.

Proposition 2.1. *Let $\varphi\colon S \longrightarrow T$ be a mapping. Then there are functors Δ_φ from $\mathbf{Set}^T$ to $\mathbf{Set}^S$ and $\coprod_\varphi$ from $\mathbf{Set}^S$ to $\mathbf{Set}^T$ such that $\coprod_\varphi \dashv \Delta_\varphi$. We write θ^φ, η^φ, and ε^φ, respectively, for the natural isomorphism, the unit, and the counit of the adjunction. Besides, there exists a contravariant functor* MSet *from* **Set** *to* **Cat** *which sends a set S to the category $\mathrm{MSet}(S) = \mathbf{Set}^S$, and a mapping φ from S to T to the functor Δ_φ from $\mathbf{Set}^T$ to $\mathbf{Set}^S$; and a pseudo-functor* $\mathrm{MSet}^{\amalg}$ *from* **Set** *to the* 2*-category* **Cat** *given by the following data*

1. *The object mapping of* $\mathrm{MSet}^{\amalg}$ *is that which sends a set S to the category $\mathrm{MSet}^{\amalg}(S) = \mathbf{Set}^S$.*
2. *The morphism mapping of* $\mathrm{MSet}^{\amalg}$ *is that which sends a mapping φ from S to T to the functor $\mathrm{MSet}^{\amalg}(\varphi) = \coprod_\varphi$ from $\mathbf{Set}^S$ to $\mathbf{Set}^T$.*
3. *For every $\varphi\colon S \longrightarrow T$ and $\psi\colon T \longrightarrow U$, the natural isomorphism $\gamma^{\varphi,\psi}$ from $\coprod_\psi \circ \coprod_\varphi$ to $\coprod_{\psi\circ\varphi}$ is that which is defined, for every S-sorted set A, as the U-sorted mapping from $\coprod_\psi(\coprod_\varphi A)$ to $\coprod_{\psi\circ\varphi} A$ that in the u-th coordinate, with $u \in U$, is $((a,s),\varphi(s)) \mapsto (a,s)$, if there exists an $s \in S$ such that $u = \psi(\varphi(s))$, and is the identity at $\varnothing$, otherwise.*
4. *For every set S, the natural isomorphism ν^S from $\mathrm{Id}_{\mathbf{Set}^S}$ to $\coprod_{\mathrm{id}_S}$ is that which is defined, for every S-sorted set A and $s \in S$, as the canonical isomorphism from A_s to $A_s \times \{s\}$.*

PROOF. Let Δ_φ be the functor from $\mathbf{Set}^T$ to $\mathbf{Set}^S$ defined as follows: its object mapping sends each T-sorted set A to the S-sorted set $A_\varphi = (A_{\varphi(s)})_{s\in S}$, i.e., the composite mapping $A\circ\varphi$; its arrow mapping sends each T-sorted mapping $f\colon A\longrightarrow B$ to the S-sorted mapping $f_\varphi = (f_{\varphi(s)})_{s\in S}\colon A_\varphi\longrightarrow B_\varphi$. Let $\coprod_\varphi$ be the functor from $\mathbf{Set}^S$ to $\mathbf{Set}^T$ defined as follows: its object mapping sends each S-sorted set A to the T-sorted set $\coprod_\varphi A = (\coprod_{s\in\varphi^{-1}[t]} A_s)_{t\in T}$; its arrow mapping sends each S-sorted mapping $f\colon A\longrightarrow B$ to the T-sorted mapping $\coprod_\varphi f = (\coprod_{s\in\varphi^{-1}[t]} f_s)_{t\in T}\colon \coprod_\varphi A\longrightarrow \coprod_\varphi B$. Then the functor $\coprod_\varphi$ is a left adjoint for Δ_φ. Indeed, let A be an S-sorted set, then the pair $(\eta^\varphi_A, \coprod_\varphi A)$, where η^φ_A is the S-sorted mapping from A to $\Delta_\varphi(\coprod_\varphi A) = (\coprod_{x\in\varphi^{-1}[\varphi(s)]} A_x)_{s\in S}$ whose s-th coordinate, for $s\in S$, is the canonical embedding from A_s to $\coprod_{x\in\varphi^{-1}[\varphi(s)]} A_x$, is a universal morphism from A to Δ_φ. This is so because, for a T-sorted set B and an S-sorted mapping $f\colon A\longrightarrow B_\varphi$, the T-sorted mapping $f^\S = (f^\S_t)_{t\in T}$ from $\coprod_\varphi A$ to B, where, for $t\in T$, $f^\S_t$ is the unique mapping $[f_s]_{s\in\varphi^{-1}[t]}$ from $\coprod_{s\in\varphi^{-1}[t]} A_s$ to $B_t = B_{\varphi(s)}$ such that, for every $s\in\varphi^{-1}[t]$, the following diagram commutes

$$\begin{tikzcd}
A_s \arrow[r, "\mathrm{in}_s"] \arrow[dr, "f_s"'] & \coprod_{s\in\varphi^{-1}[t]} A_s \arrow[d, "{[f_s]_{s\in\varphi^{-1}[t]} = f^\S_t}"] \\
 & B_t = B_{\varphi(s)}
\end{tikzcd}$$

is such that $f = \Delta_\varphi(f^\S)\circ\eta^\varphi_A$ and unique with such a property.

Otherwise stated, $\coprod_\varphi A$ is, simply, $\mathrm{Lan}_\varphi A$, i.e., the left Kan extension of A along φ, recalling that every set is the set of objects of a discrete category and every mapping between sets the object mapping of a functor between discrete categories.

Let us recall that, for each S-sorted set X and each T-sorted set A,

$$\theta^\varphi_{X,A}\colon \mathrm{Hom}_{\mathbf{Set}^T}(\textstyle\coprod_\varphi X, A)\longrightarrow \mathrm{Hom}_{\mathbf{Set}^S}(X, A_\varphi),$$

the value of the natural isomorphism θ^φ at (X, A), can be expressed in terms of η^φ_X, for each T-sorted mapping $f\colon \coprod_\varphi X\longrightarrow A$, as

$$\theta^\varphi_{X,A}(f) = f_\varphi\circ\eta^\varphi_X\colon X\longrightarrow A_\varphi.$$

To prove that $\mathrm{MSet}^{\amalg}$ is a pseudo-functor, it is enough to verify the coherence axioms. But given the situation

$$S \xrightarrow{\ \varphi\ } T \xrightarrow{\ \psi\ } U \xrightarrow{\ \xi\ } X,$$

the following diagrams commute

$$
\begin{tikzcd}
\coprod_{\xi} \circ \coprod_{\psi} \circ \coprod_{\varphi} \arrow[r, "\mathrm{id}_{\coprod_{\xi}} * \gamma^{\varphi,\psi}"] \arrow[d, "\gamma^{\psi,\xi} * \mathrm{id}_{\coprod_{\varphi}}"'] & \coprod_{\xi} \circ \coprod_{\psi\circ\varphi} \arrow[d, "\gamma^{\psi\circ\varphi,\xi}"] \\
\coprod_{\xi\circ\psi} \circ \coprod_{\varphi} \arrow[r, "\gamma^{\varphi,\xi\circ\psi}"'] & \coprod_{\xi\circ\psi\circ\varphi}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\coprod_{\varphi} \circ \mathrm{Id}_{\mathbf{Set}^S} \arrow[r, "\mathrm{id}_{\coprod_{\varphi}} * \nu^S"] \arrow[d, "\mathrm{id}_{\coprod_{\varphi}}"'] & \coprod_{\varphi} \circ \coprod_{\mathrm{id}_S} \arrow[d, "\gamma^{\mathrm{id}_S,\varphi}"] \\
\coprod_{\varphi} \arrow[r, "\mathrm{id}_{\coprod_{\varphi}}"'] & \coprod_{\varphi\circ\mathrm{id}_S}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Id}_{\mathbf{Set}^T} \circ \coprod_{\varphi} \arrow[r, "\nu^T * \mathrm{id}_{\coprod_{\varphi}}"] \arrow[d, "\mathrm{id}_{\coprod_{\varphi}}"'] & \coprod_{\mathrm{id}_T} \circ \coprod_{\varphi} \arrow[d, "\gamma^{\varphi,\mathrm{id}_T}"] \\
\coprod_{\varphi} \arrow[r, "\mathrm{id}_{\coprod_{\varphi}}"'] & \coprod_{\mathrm{id}_T\circ\varphi}
\end{tikzcd}
$$

■

Henceforth, when dealing with a pseudo-functor we will restrict ourselves to define explicitly only its object and morphism mappings, if about the remaining data and conditions there is not any doubt.

By applying the EG-construction to MSet we obtain the category of many-sorted sets as stated in the following definition.

Definition 2.2. The category **MSet**, of *many-sorted sets* and *many-sorted mappings*, is given by $\mathbf{MSet} = \int^{\mathbf{Set}} \mathrm{MSet}$. Therefore **MSet** has as objects the pairs (S, A), where S is a set and A an S-sorted set, and as morphisms from (S, A) to (T, B) the pairs (φ, f), where $\varphi\colon S \longrightarrow T$ and $f\colon A \longrightarrow B_{\varphi}$.

Our next goal is to define the category **Sig**. But before doing that we agree that, for a set of sorts S in $\boldsymbol{\mathcal{U}}$, $\mathbf{Sig}(S)$ denotes the category of S-sorted signatures and S-sorted signature morphisms, i.e., the category $\mathbf{Set}^{S^\star \times S}$, where $S^\star$ is the underlying set of the free monoid on S. Therefore an S-*sorted signature* is a function Σ from $S^\star \times S$ to $\boldsymbol{\mathcal{U}}$ which sends a pair $(w, s) \in S^\star \times S$ to the set $\Sigma_{w,s}$ of the *formal operations* of *arity* w, *sort* (or *coarity*) s, and *biarity* (w, s); and an S-*sorted signature morphism* from Σ to Σ' is an ordered triple (Σ, d, Σ'), written as $d\colon \Sigma \longrightarrow \Sigma'$, where $d = (d_{w,s})_{(w,s)\in S^\star\times S} \in \prod_{(w,s)\in S^\star\times S} \mathrm{Hom}(\Sigma_{w,s}, \Sigma'_{w,s})$. Thus, for every $(w, s) \in S^\star \times S$, $d_{w,s}$ is a mapping from $\Sigma_{w,s}$ to $\Sigma'_{w,s}$ which sends a formal operation σ in $\Sigma_{w,s}$ to the formal operation $d_{w,s}(\sigma)$ ($d(\sigma)$ for short) in $\Sigma'_{w,s}$.

PROPOSITION 2.3. *There exists a contravariant functor* Sig *from* **Set** *to* **Cat**. *Its object mapping sends each set of sorts S to* $\mathrm{Sig}(S) = \mathbf{Sig}(S)$*; its arrow mapping sends each mapping φ from S to T to the functor* $\mathrm{Sig}(\varphi) = \Delta_{\varphi^\star \times \varphi}$ *from* $\mathbf{Sig}(T)$ *to* $\mathbf{Sig}(S)$ *which relabels T-sorted signatures into S-sorted signatures, i.e.,* $\mathrm{Sig}(\varphi)$ *assigns to a T-sorted signature Λ the S-sorted signature* $\mathrm{Sig}(\varphi)(\Lambda) = \Lambda_{\varphi^\star \times \varphi}$*, and assigns to a morphism of T-sorted signatures d from Λ to Λ' the morphism of S-sorted signatures* $\mathrm{Sig}(\varphi)(d) = d_{\varphi^\star \times \varphi}$ *from* $\Lambda_{\varphi^\star \times \varphi}$ *to* $\Lambda'_{\varphi^\star \times \varphi}$.

By applying the EG-construction to Sig we obtain the category of many-sorted signatures as stated in the following definition.

DEFINITION 2.4. The category **Sig**, of *many-sorted signatures* and *many-sorted signature morphisms*, is given by $\mathbf{Sig} = \int^{\mathbf{Set}} \mathrm{Sig}$. Therefore **Sig** has as objects the pairs (S, Σ), where S is a set of sorts and Σ an S-sorted signature and as many-sorted signature morphisms from (S, Σ) to (T, Λ) the pairs (φ, d), where $\varphi\colon S \longrightarrow T$ is a morphism in **Set** while $d\colon \Sigma \longrightarrow \Lambda_{\varphi^\star \times \varphi}$ is a morphism in $\mathbf{Sig}(S)$. The composition of $(\varphi, d)\colon (S, \Sigma) \longrightarrow (T, \Lambda)$ and $(\psi, e)\colon (T, \Lambda) \longrightarrow (U, \Omega)$, denoted by $(\psi, e) \circ (\varphi, d)$, is $(\psi \circ \varphi, e_{\varphi^\star \times \varphi} \circ d)$, where $e_{\varphi^\star \times \varphi}\colon \Lambda_{\varphi^\star \times \varphi} \longrightarrow (\Omega_{\psi^\star \times \psi})_{\varphi^\star \times \varphi} (= \Omega_{(\psi \circ \varphi)^\star \times (\psi \circ \varphi)})$. Henceforth, unless otherwise stated, we will write $\mathbf{\Sigma}$, $\mathbf{\Lambda}$, and $\mathbf{\Omega}$ instead of (S, Σ), (T, Λ), and (U, Ω), respectively, and $\mathbf{d}$ and $\mathbf{e}$ instead of (φ, d) and (ψ, e), respectively. Furthermore, to shorten terminology, we will say *signature* and *signature morphism* instead of *many-sorted signature* and *many-sorted signature morphism*, respectively.

Since it will be used afterwards we introduce, for a signature $\mathbf{\Sigma}$, an S-sorted set A, an S-sorted mapping f from A to B, and a word w on S, i.e., an element w of $S^\star$, the following notation and terminology. We write A_w for $\prod_{i \in |w|} A_{w_i}$, where, we recall, $|w|$ is the length of w, and f_w for the mapping $\prod_{i \in |w|} f_{w_i} = \left\langle f_{w_i} \circ \mathrm{pr}_{w_i} \right\rangle_{i \in |w|}$ from A_w to B_w which sends $(a_i)_{i \in |w|}$ in A_w to $(f_{w_i}(a_i))_{i \in |w|}$ in B_w, where, for each $i \in |w|$, pr_{w_i} is the canonical projection from A_w to A_{w_i}. Moreover, we let $\mathcal{O}_S(A)$ stand for the $S^\star \times S$-sorted set $(\mathrm{Hom}(A_w, A_s))_{(w,s) \in S^\star \times S}$ and we call it the $S^\star \times S$-sorted set of the *finitary operations on A*.

We next turn to defining the category **Alg** of many-sorted algebras. But before doing that we agree that, for a signature $\mathbf{\Sigma}$, $\mathbf{Alg}(\mathbf{\Sigma})$ denotes the category of $\mathbf{\Sigma}$-algebras (and $\mathbf{\Sigma}$-homomorphisms). By a $\mathbf{\Sigma}$-*algebra* is meant a pair $\mathbf{A} = (A, F)$, where A is an S-sorted set and F a $\mathbf{\Sigma}$-*algebra structure on A*, i.e., a morphism $F = (F_{w,s})_{(w,s) \in S^\star \times S}$ in $\mathbf{Sig}(S)$ from Σ to $\mathcal{O}_S(A)$ (for a pair $(w, s) \in S^\star \times S$ and a $\sigma \in \Sigma_{w,s}$, to simplify notation we let F_σ

stand for $F_{w,s}(\sigma)$). Sometimes, to avoid any confusion, we will denote the $\mathbf{\Sigma}$-algebra structure of a $\mathbf{\Sigma}$-algebra $\mathbf{A}$ by $F^{\mathbf{A}}$ and the components of $F^{\mathbf{A}}$ as $F^{\mathbf{A}}_{\sigma}$. A $\mathbf{\Sigma}$-*homomorphism* from a $\mathbf{\Sigma}$-algebra $\mathbf{A}$ to another $\mathbf{B} = (B, G)$, is a triple $(\mathbf{A}, f, \mathbf{B})$, written as $f\colon \mathbf{A} \longrightarrow \mathbf{B}$, where f is an S-sorted mapping from A to B that preserves the structure in the sense that, for every (w, s) in $S^{\star} \times S$, every σ in $\Sigma_{w,s}$, and every $(a_i)_{i\in|w|}$ in A_w, it happens that

$$f_s(F_{\sigma}((a_i)_{i\in|w|})) = G_{\sigma}(f_w((a_i)_{i\in|w|})).$$

Proposition 2.5. *There exists a contravariant functor* Alg *from* **Sig** *to* **Cat**. *Its object mapping sends each signature* $\mathbf{\Sigma}$ *to* $\mathrm{Alg}(\mathbf{\Sigma}) = \mathbf{Alg}(\mathbf{\Sigma})$, *the category of* $\mathbf{\Sigma}$-*algebras; its arrow mapping sends each signature morphism* $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ *to the functor* $\mathrm{Alg}(\mathbf{d}) = \mathbf{d}^{*}\colon \mathbf{Alg}(\mathbf{\Lambda}) \longrightarrow \mathbf{Alg}(\mathbf{\Sigma})$ *defined as follows: its object mapping sends each* $\mathbf{\Lambda}$-*algebra* $\mathbf{B} = (B, G)$ *to the* $\mathbf{\Sigma}$-*algebra* $\mathbf{d}^{*}(\mathbf{B}) = (B_{\varphi}, G^{\mathbf{d}})$, *where* $G^{\mathbf{d}}$ *is the composition of the* $S^{\star} \times S$-*sorted mappings* d *from* Σ *to* $\Lambda_{\varphi^{\star}\times\varphi}$ *and* $G_{\varphi^{\star}\times\varphi}$ *from* $\Lambda_{\varphi^{\star}\times\varphi}$ *to* $\mathcal{O}_T(B)_{\varphi^{\star}\times\varphi}$ *(for* $\sigma \in \Sigma_{w,s}$, *to shorten notation, we let* $G_{d(\sigma)}$ *stand for the value of* $G^{\mathbf{d}}$ *at* σ*); its arrow mapping sends each* $\mathbf{\Lambda}$-*homomorphism* f *from* $\mathbf{B}$ *to* $\mathbf{B}'$ *to the* $\mathbf{\Sigma}$-*homomorphism* $\mathbf{d}^{*}(f) = f_{\varphi}$ *from* $\mathbf{d}^{*}(\mathbf{B})$ *to* $\mathbf{d}^{*}(\mathbf{B}')$.

Proof. For every $\mathbf{\Lambda}$-algebra $\mathbf{B} = (B, G)$ it is the case that G is a morphism from Λ to $\mathcal{O}_T(B)$. Then, by composing d and $G_{\varphi^{\star}\times\varphi}$, and taking into account that $\mathcal{O}_T(B)_{\varphi^{\star}\times\varphi}$ is identical to $\mathcal{O}_S(B_{\varphi})$, we infer that $G^{\mathbf{d}} = G_{\varphi^{\star}\times\varphi} \circ d$ is a $\mathbf{\Sigma}$-algebra structure on the S-sorted set B_{φ}. On the other hand, for every (w, s) in $S^{\star} \times S$ and every $\sigma \in \Sigma_{w,s}$, it happens that $d(\sigma) \in \Lambda_{\varphi^{\star}(w),\varphi(s)}$. Thus, f being a $\mathbf{\Lambda}$-homomorphism from (B, G) to (B', G'), we infer that $f_{\varphi(s)} \circ G_{d(\sigma)} = G'_{d(\sigma)} \circ f_{\varphi^{\star}(w)}$. Hence $(f_{\varphi})_s \circ G^{\mathbf{d}}_{\sigma} = G'^{\mathbf{d}}_{\sigma} \circ (f_{\varphi})_w$, since $G^{\mathbf{d}}_{\sigma} = G_{d(\sigma)}$ and $G'^{\mathbf{d}}_{\sigma} = G'_{d(\sigma)}$. Therefore f_{φ} is a $\mathbf{\Sigma}$-homomorphism from $(B_{\varphi}, G^{\mathbf{d}})$ to $(B'_{\varphi}, G'^{\mathbf{d}})$.

Since identities and composites are, obviously, preserved by $\mathbf{d}^{*}$, it follows that $\mathbf{d}^{*}$ is a functor from $\mathbf{Alg}(\mathbf{\Lambda})$ to $\mathbf{Alg}(\mathbf{\Sigma})$. ■

By applying the EG-construction to Alg we obtain the category of many-sorted algebras as stated in the following definition.

Definition 2.6. The category **Alg**, of *many-sorted algebras* and *many-sorted algebra homomorphisms*, is given by $\mathbf{Alg} = \int^{\mathbf{Sig}} \mathrm{Alg}$. Therefore the category **Alg** has as objects the pairs $(\mathbf{\Sigma}, \mathbf{A})$, where $\mathbf{\Sigma}$ is a signature and $\mathbf{A}$ a $\mathbf{\Sigma}$-algebra, and as morphisms from $(\mathbf{\Sigma}, \mathbf{A})$ to $(\mathbf{\Lambda}, \mathbf{B})$, the pairs $(\mathbf{d}, f)$, with $\mathbf{d}$ a signature morphism from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$ and f a $\mathbf{\Sigma}$-homomorphism from $\mathbf{A}$ to $\mathbf{d}^{*}(\mathbf{B})$. Henceforth, to shorten terminology, we will say *algebra* and *algebra* *homomorphism*, or, simply, *homomorphism*, instead of *many-sorted algebra* and *many-sorted algebra homomorphism*, respectively.

PROPOSITION 2.7. *The category* **Alg** *is a concrete and uniquely transportable category.*

PROOF. It is enough to specify a functor from **Alg** to a convenient category of sorted sets labelled by signatures.

Let $\mathrm{G}_{\mathbf{MSet}}$ be the forgetful functor from **Alg** to **MSet** (that is not a fibration), π_{Alg} the projection functor for **Alg**, and $(\mathbf{MSet}\times_{\mathbf{Set}}\mathbf{Sig}, (\mathrm{P}_0, \mathrm{P}_1))$ the pullback of the projection functors π_{MSet} and π_{Sig}, for **MSet** and **Sig**, respectively, where

1. the category $\mathbf{MSet}\times_{\mathbf{Set}}\mathbf{Sig}$ has as objects, essentially, triples (S, Σ, A), with (S, Σ) a signature and A an S-sorted set, and as morphisms from (S, Σ, A) to (T, Λ, B) triples (φ, d, f), such that (φ, d) is a signature morphism from (S, Σ) to (T, Λ) and (φ, f) a mapping from (S, A) to (T, B), while
2. P_0 is the functor from $\mathbf{MSet}\times_{\mathbf{Set}}\mathbf{Sig}$ to **MSet** which sends a morphism (φ, d, f) from (S, Σ, A) to (T, Λ, B) to the morphism (φ, f) from (S, A) to (T, B), and P_1 is the functor from $\mathbf{MSet}\times_{\mathbf{Set}}\mathbf{Sig}$ to **Sig** which sends a morphism (φ, d, f) from (S, Σ, A) to (T, Λ, B) to the signature morphism (φ, d) from (S, Σ) to (T, Λ).

Then we have that the structural functors P_0 and P_1 are fibrations (the proof is straightforward), and that the unique functor $\mathrm{G}\colon \mathbf{Alg} \longrightarrow \mathbf{MSet}\times_{\mathbf{Set}}\mathbf{Sig}$ such that $\mathrm{P}_0 \circ \mathrm{G} = \mathrm{G}_{\mathbf{MSet}}$ and $\mathrm{P}_1 \circ \mathrm{G} = \pi_{\mathrm{Alg}}$, as in the following diagram

$$
\begin{tikzcd}
\mathbf{Alg} \arrow[dr, "\mathrm{G}"] \arrow[drr, bend left, "\pi_{\mathrm{Alg}}"] \arrow[ddr, bend right, "\mathrm{G}_{\mathbf{MSet}}"'] & & \\
& \mathbf{MSet}\times_{\mathbf{Set}}\mathbf{Sig} \arrow[r, "\mathrm{P}_1"] \arrow[d, "\mathrm{P}_0"'] & \mathbf{Sig} \arrow[d, "\pi_{\mathrm{Sig}}"] \\
& \mathbf{MSet} \arrow[r, "\pi_{\mathrm{MSet}}"'] & \mathbf{Set}
\end{tikzcd}
$$

makes the category **Alg** a concrete and uniquely transportable category on the category $\mathbf{MSet}\times_{\mathbf{Set}}\mathbf{Sig}$. ∎

Before we prove the existence of a left adjoint $\mathbf{T}$ to the functor G from **Alg** to $\mathbf{MSet}\times_{\mathbf{Set}}\mathbf{Sig}$, we agree on the following notation and terminology.

For a signature $\mathbf{\Sigma}$ in **Sig**, the functor $\mathbf{T}_{\mathbf{\Sigma}}$ from $\mathbf{Set}^S$ to $\mathbf{Alg}(\mathbf{\Sigma})$ is the left adjoint to the forgetful functor $\mathrm{G}_{\mathbf{\Sigma}}$ from $\mathbf{Alg}(\mathbf{\Sigma})$ to $\mathbf{Set}^S$. For a signature $\mathbf{\Sigma}$ and an S-sorted set of variables X, $\mathbf{T}_{\mathbf{\Sigma}}(X)$ is the *free* (also called the *term* or *word*) *$\mathbf{\Sigma}$-algebra on X*, and η_X is the *insertion (of the generators) X into* $\mathrm{T}_{\mathbf{\Sigma}}(X)$, the underlying S-sorted set of $\mathbf{T}_{\mathbf{\Sigma}}(X)$. For the convenience of the reader we next recall the construction of $\mathbf{T}_{\mathbf{\Sigma}}(X)$. For a signature $\mathbf{\Sigma}$ in **Sig** and an S-sorted set (of variables) X, we define a $\mathbf{\Sigma}$-algebra $\mathbf{W}_{\mathbf{\Sigma}}(X)$, the *$\mathbf{\Sigma}$-algebra of $\mathbf{\Sigma}$-rows* in X, as follows: $\mathrm{W}_{\mathbf{\Sigma}}(X) = (\mathrm{W}_{\mathbf{\Sigma}}(X)_s)_{s\in S}$, the underlying S-sorted set of $\mathbf{W}_{\mathbf{\Sigma}}(X)$, is $((\coprod\Sigma \amalg \coprod X)^\star)_{s\in S}$, i.e., for every s in S, $\mathrm{W}_{\mathbf{\Sigma}}(X)_s = (\coprod\Sigma \amalg \coprod X)^\star$, where $(\coprod\Sigma \amalg \coprod X)^\star$ is the underlying set of the free monoid on $\coprod\Sigma \amalg \coprod X$. On the S-sorted set $\mathrm{W}_{\mathbf{\Sigma}}(X)$ we define a $\mathbf{\Sigma}$-algebra structure by concatenation; thus, for every $(w,s) \in S^\star \times S$ and every $\sigma \in \Sigma_{w,s}$, F_σ, the structural operation associated to σ, is the mapping from $\mathrm{W}_{\mathbf{\Sigma}}(X)_w$ to $\mathrm{W}_{\mathbf{\Sigma}}(X)_s$, i.e., from $((\coprod\Sigma \amalg \coprod X)^\star)^{|w|}$ to $(\coprod\Sigma \amalg \coprod X)^\star$, which sends $(P_i)_{i\in|w|}$ in $((\coprod\Sigma\amalg\coprod X)^\star)^{|w|}$ to $(\sigma) \curlywedge \curlywedge_{i\in|w|} P_i$ in $(\coprod\Sigma\amalg\coprod X)^\star$, i.e., to the concatenation of (σ) and the concatenation of the words P_i in the family $(P_i)_{i\in|w|}$, where (σ) is the image of σ under the canonical embeddings from $\Sigma_{w,s}$ to $(\coprod\Sigma \amalg \coprod X)^\star$. Then $\mathbf{T}_{\mathbf{\Sigma}}(X)$, the free $\mathbf{\Sigma}$-algebra on X, is the subalgebra de $\mathbf{W}_{\mathbf{\Sigma}}(X)$ generated by the S-sorted set $(\{(x) \mid x \in X_s\})_{s\in S}$, where, for every $s \in S$ and every $x \in X_s$, (x) is the image of x under the canonical embeddings from X_s to $(\coprod\Sigma \amalg \coprod X)^\star$.

The following diagrams show the aforementioned canonical embeddings from X_s, resp., $\Sigma_{w,s}$, to $\mathrm{W}_{\mathbf{\Sigma}}(X)_s$:

$$
\begin{tikzcd}
X_s \arrow[r, "\mathrm{in}_{X_s}"] & \coprod X \arrow[r, "\mathrm{in}_{\coprod X}"] & \coprod\Sigma\amalg\coprod X \arrow[r, "\eta_{\coprod\Sigma\amalg\coprod X}"] & (\coprod\Sigma\amalg\coprod X)^\star \\
x \arrow[r, mapsto] & (x,s) \arrow[r, mapsto] & ((x,s),1) \arrow[r, mapsto] & (((x,s),1)) \equiv (x)
\end{tikzcd}
$$

$$
\begin{tikzcd}
\Sigma_{w,s} \arrow[r, "\mathrm{in}_{\Sigma_{w,s}}"] & \coprod\Sigma \arrow[r, "\mathrm{in}_{\coprod\Sigma}"] & \coprod\Sigma\amalg\coprod X \arrow[r, "\eta_{\coprod\Sigma\amalg\coprod X}"] & (\coprod\Sigma\amalg\coprod X)^\star \\
\sigma \arrow[r, mapsto] & (\sigma,(w,s)) \arrow[r, mapsto] & ((\sigma,(w,s)),0) \arrow[r, mapsto] & (((\sigma,(w,s)),0)) \equiv (\sigma)
\end{tikzcd}
$$

For a $\mathbf{\Sigma}$-algebra $\mathbf{A}$ and a *valuation f of the S-sorted set of variables X in A*, i.e., an S-sorted mapping f from X to A, we will denote by $f^\sharp$ the *canonical extension of f to* $\mathbf{T}_{\mathbf{\Sigma}}(X)$, i.e., the unique $\mathbf{\Sigma}$-homomorphism from $\mathbf{T}_{\mathbf{\Sigma}}(X)$ to $\mathbf{A}$ such that $f^\sharp \circ \eta_X = f$. For an S-sorted mapping f from X to Y, we will denote by $f^@$ the unique $\mathbf{\Sigma}$-homomorphism from $\mathbf{T}_{\mathbf{\Sigma}}(X)$ to $\mathbf{T}_{\mathbf{\Sigma}}(Y)$ such that $f^@ \circ \eta_X = \eta_Y \circ f$, i.e., the value of the functor $\mathbf{T}_{\mathbf{\Sigma}}$ at f. Therefore $f^@$ is also $(\eta_Y \circ f)^\sharp$. Moreover, by transposing to the many-sorted case the terminology coined for the single-sorted case, we call, for $s \in S$, the elements of $\mathrm{T}_{\mathbf{\Sigma}}(X)_s$, *many-sorted terms for* $\mathbf{\Sigma}$ *of type* (X, s), henceforth abbreviated to *terms for* $\mathbf{\Sigma}$ *of type* (X, s), or, simply, to *terms of type* (X, s).

The functor $\mathbf{T}$, which will be obtained from the family $(\mathbf{T}_{\mathbf{\Sigma}})_{\mathbf{\Sigma} \in \mathbf{Sig}}$, allow us to obtain translations between free algebras and will be used, after defining in the third section the many-sorted equations, to translate, for a signature morphism, many-sorted equations for the source signature to many-sorted equations for the target signature. This translation of equations, together with the invariant character of the relation of satisfaction under change of notation, will allows us to define, also in the third section, the many-sorted equational institution (which is more general than that defined by Goguen and Burstall in [15] and, in addition, embodies the essentials of semantical many-sorted equational deduction).

PROPOSITION 2.8. *There exists a functor* $\mathbf{T}\colon \mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig} \longrightarrow \mathbf{Alg}$ *left adjoint to the functor* G *from* $\mathbf{Alg}$ *to* $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$.

PROOF. The functor $\mathbf{T}$ from $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$ to $\mathbf{Alg}$ defined on objects (S, Σ, X) by $\mathbf{T}(S, \Sigma, X) = (\mathbf{\Sigma}, \mathbf{T}_{\mathbf{\Sigma}}(X))$ and on arrows (φ, d, f) from (S, Σ, X) to (T, Λ, Y) by $\mathbf{T}(\varphi, d, f) = (\mathbf{d}, f^{\mathbf{d}})\colon (\mathbf{\Sigma}, \mathbf{T}_{\mathbf{\Sigma}}(X)) \longrightarrow (\mathbf{\Lambda}, \mathbf{T}_{\mathbf{\Lambda}}(Y))$, where $f^{\mathbf{d}} = ((\eta_Y)_\varphi \circ f)^\sharp$ is the canonical extension of the S-sorted mapping $(\eta_Y)_\varphi \circ f$ from X to $\mathrm{T}_{\mathbf{\Lambda}}(Y)_\varphi$ to the free $\mathbf{\Sigma}$-algebra on X, is left adjoint to the functor G. ∎

For a morphism $(\varphi, d, f)\colon (S, \Sigma, X) \longrightarrow (T, \Lambda, Y)$ in $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$, the functor $\mathbf{T}$ acting on (φ, d, f) allows us to obtain the $\mathbf{\Sigma}$-homomorphism $f^{\mathbf{d}}$ from $\mathbf{T}_{\mathbf{\Sigma}}(X)$ to $\mathbf{T}_{\mathbf{\Lambda}}(Y)_\varphi$. Hence, for each $s \in S$, $f^{\mathbf{d}}_s$ translates terms for $\mathbf{\Sigma}$ of type (X, s) into terms for $\mathbf{\Lambda}$ of type $(Y, \varphi(s))$. In particular, the unit η^φ of the adjunction $\coprod_\varphi \dashv \Delta_\varphi$ provides, for each S-sorted set X, the S-sorted mapping $\eta^\varphi_X\colon X \longrightarrow (\coprod_\varphi X)_\varphi$ and if $\mathbf{d}$ is a morphism of signatures from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$, then $(\varphi, d, \eta^\varphi_X)\colon (S, \Sigma, X) \longrightarrow (T, \Lambda, \coprod_\varphi X)$ is a morphism in $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$. Hence the functor $\mathbf{T}$ acting on $(\varphi, d, \eta^\varphi_X)$ determines the morphism $(\mathbf{d}, \eta^{\mathbf{d}}_X)$ from $(\mathbf{\Sigma}, \mathbf{T}_{\mathbf{\Sigma}}(X))$ to $(\mathbf{\Lambda}, \mathbf{T}_{\mathbf{\Lambda}}(\coprod_\varphi X))$, where $\eta^{\mathbf{d}}_X = ((\eta_{\coprod_\varphi X})_\varphi \circ \eta^\varphi_X)^\sharp$ is the $\mathbf{\Sigma}$-homomorphism from $\mathbf{T}_{\mathbf{\Sigma}}(X)$ to $\mathbf{T}_{\mathbf{\Lambda}}(\coprod_\varphi X)_\varphi$ that extends the S-sorted mapping $(\eta_{\coprod_\varphi X})_\varphi \circ \eta^\varphi_X$ from X to $\mathrm{T}_{\mathbf{\Lambda}}(\coprod_\varphi X)_\varphi$. Therefore, for every $s \in S$, $\eta^{\mathbf{d}}_{X,s}$ translates terms for $\mathbf{\Sigma}$ of type (X, s) into terms for $\mathbf{\Lambda}$ of type $(\coprod_\varphi X, \varphi(s))$. The $\mathbf{\Sigma}$-homomorphisms $\eta^{\mathbf{d}}_X$, as stated in the following proposition, are to be more precise the components of a natural transformation, and this contributes to explain their relevance as *translators*.

Proposition 2.9. *Let* $\mathbf{d} = (\varphi, d)$ *be a morphism of signatures from* $\mathbf{\Sigma} = (S, \Sigma)$ *to* $\mathbf{\Lambda} = (T, \Lambda)$. *Then the family* $\eta^{\mathbf{d}} = (\eta^{\mathbf{d}}_X)_{X \in \boldsymbol{\mathcal{U}}}$, *which to an* S*-sorted set* X *assigns the* $\mathbf{\Sigma}$*-homomorphism* $\eta^{\mathbf{d}}_X$ *from* $\mathbf{T}_{\mathbf{\Sigma}}(X)$ *to* $\mathbf{T}_{\mathbf{\Lambda}}(\coprod_\varphi X)_\varphi$, *is a natural transformation from* $\mathbf{T}_{\mathbf{\Sigma}}$ *to* $\mathbf{d}^* \circ \mathbf{T}_{\mathbf{\Lambda}} \circ \coprod_\varphi$, *and so, for the forgetful functor* $\mathrm{G}_{\mathbf{\Sigma}}$ *from* $\mathbf{Alg}(\mathbf{\Sigma})$ *to* $\mathbf{Set}^S$, *the family* $\mathrm{G}_{\mathbf{\Sigma}} * \eta^{\mathbf{d}}$, *i.e., the horizontal composition of the natural transformations* $\eta^{\mathbf{d}}$ *and* $\mathrm{id}_{\mathrm{G}_{\mathbf{\Sigma}}}$, *also denoted by* $\eta^{\mathbf{d}}$, *is a natural transformation from* $\mathrm{T}_{\mathbf{\Sigma}} = \mathrm{G}_{\mathbf{\Sigma}} \circ \mathbf{T}_{\mathbf{\Sigma}}$ *to* $\Delta_\varphi \circ \mathrm{T}_{\mathbf{\Lambda}} \circ \coprod_\varphi$ *(see the diagram below), taking into account that* $\mathrm{G}_{\mathbf{\Sigma}} \circ \mathbf{d}^* = \Delta_\varphi \circ \mathrm{G}_{\mathbf{\Lambda}}$ *and* $\mathrm{T}_{\mathbf{\Lambda}} = \mathrm{G}_{\mathbf{\Lambda}} \circ \mathbf{T}_{\mathbf{\Lambda}}$.

$$
\begin{tikzcd}
\mathbf{Set}^S \arrow[rrr, bend left, "\mathbf{T}_{\mathbf{\Sigma}}"] \arrow[dr, "\coprod_\varphi"'] & & & \mathbf{Alg}(\mathbf{\Sigma}) \arrow[r, bend left, "\mathrm{G}_{\mathbf{\Sigma}}"] \arrow[r, bend right, "\mathrm{G}_{\mathbf{\Sigma}}"'] \arrow[r, phantom, "\Downarrow \mathrm{id}_{\mathrm{G}_{\mathbf{\Sigma}}}"] & \mathbf{Set}^S \\
 & \mathbf{Set}^T \arrow[r, "\mathbf{T}_{\mathbf{\Lambda}}"'] \arrow[u, phantom, "\Downarrow \eta^{\mathbf{d}}"] & \mathbf{Alg}(\mathbf{\Lambda}) \arrow[ur, "\mathbf{d}^*"'] & &
\end{tikzcd}
$$

$$
= \begin{tikzcd}
\mathbf{Set}^S \arrow[rr, bend left, "\mathrm{G}_{\mathbf{\Sigma}} \circ \mathbf{T}_{\mathbf{\Sigma}} = \mathrm{T}_{\mathbf{\Sigma}}"] \arrow[rr, bend right, "\mathrm{G}_{\mathbf{\Sigma}} \circ \mathbf{d}^* \circ \mathbf{T}_{\mathbf{\Lambda}} \circ \coprod_\varphi = \Delta_\varphi \circ \mathrm{T}_{\mathbf{\Lambda}} \circ \coprod_\varphi"'] \arrow[rr, phantom, "\Downarrow \mathrm{G}_{\mathbf{\Sigma}} * \eta^{\mathbf{d}}"] & & \mathbf{Set}^S
\end{tikzcd}
$$

Proof. It follows after the commutativity of the following diagram

$$
\begin{tikzcd}
 & \mathrm{T}_{\mathbf{\Sigma}}(X) \arrow[rr, "f^{@}"] \arrow[dd, "\eta^{\mathbf{d}}_X"] & & \mathrm{T}_{\mathbf{\Sigma}}(Y) \arrow[dd, "\eta^{\mathbf{d}}_Y"] \\
X \arrow[ur, "\eta_X"] \arrow[rr, "f"] \arrow[dd, "\eta^\varphi_X"'] & & Y \arrow[ur, "\eta_Y"] \arrow[dd, "\eta^\varphi_Y"'] & \\
 & \mathrm{T}_{\mathbf{\Lambda}}(\coprod_\varphi X)_\varphi \arrow[rr, "(\coprod_\varphi f)^{@}_\varphi"] & & \mathrm{T}_{\mathbf{\Lambda}}(\coprod_\varphi Y)_\varphi \\
(\coprod_\varphi X)_\varphi \arrow[ur, "(\eta_{\coprod_\varphi X})_\varphi"] \arrow[rr, "(\coprod_\varphi f)_\varphi"] & & (\coprod_\varphi Y)_\varphi \arrow[ur, "(\eta_{\coprod_\varphi Y})_\varphi"] &
\end{tikzcd}
$$

■

The contravariant functor Alg from **Sig** to **Cat** is not only useful to construct the category **Alg**. Actually, as we will show from here to the end of this section, Alg, together with a pseudo-functor Ter from **Sig** to **Cat**, and a pseudo-extranatural transformation (Tr, θ) (from a pseudo-functor on $\mathbf{Sig}^{\mathrm{op}} \times \mathbf{Sig}$ to **Cat**, induced by Alg and Ter, to the functor $\mathrm{K}_{\mathbf{Set}}$, between the same categories, which picks **Set**), will enable us to construct a new institution on **Set**, the many-sorted term institution, denoted by $\mathfrak{Tm} = (\mathbf{Sig}, \mathrm{Alg}, \mathrm{Ter}, (\mathrm{Tr}, \theta))$, but for a concept of institution that is *strictly* more general than that of generalized **V**-institution defined by Goguen and Burstall in [15].

For the institution $\mathfrak{Tm}$ on **Set**, as we will prove, it happens that the existence of the pseudo-functor Ter follows from the fact that, for every signature $\mathbf{\Sigma}$, the terms for $\mathbf{\Sigma}$, understood in a generalized sense to be explained below, have a categorical interpretation as the morphisms of a suitable category $\mathbf{Ter}(\mathbf{\Sigma})$. Furthermore, the component Tr of the pseudo-extranatural transformation (Tr, θ) depends for its existence on the fact that the generalized terms have canonically associated generalized term operations on the algebras. Therefore, to proceed properly, we should begin by defining, for a $\mathbf{\Sigma}$-algebra $\mathbf{A}$ and an S-sorted set X, the concepts of many-sorted X-ary operation on $\mathbf{A}$ and of many-sorted X-ary term operation on $\mathbf{A}$, and the procedure of realization of terms P of type (X, s) as term operations $P^{\mathbf{A}}$ on $\mathbf{A}$.

Definition 2.10. Let X be an S-sorted set, $\mathbf{A}$ a $\mathbf{\Sigma}$-algebra, s a sort in S and $P \in \mathrm{T}_{\mathbf{\Sigma}}(X)_s$ a term for $\mathbf{\Sigma}$ of type (X, s). Then the $\mathbf{\Sigma}$-algebra of the *many-sorted X-ary operations on* $\mathbf{A}$, $\mathbf{O}_X(\mathbf{A})$, is $\mathbf{A}^{A_X}$, i.e., the direct A_X-power of $\mathbf{A}$, where A_X is $\mathrm{Hom}(X, A)$, the (ordinary) set of the S-sorted mappings from X to A. We recall that in the $\mathbf{\Sigma}$-algebra $\mathbf{O}_X(\mathbf{A})$, for every $(w, s) \in S^{\star} \times S$ and every $\sigma \in \Sigma_{w,s}$, the structural operation F_σ is a mapping from $(A^{A_X})_w = \prod_{i \in |w|} A_{w_i}^{A_X}$ to A_s^X. We next turn to explicitly define F_σ. Let $(P_i)_{i \in |w|}$ be a family in $(A^{A_X})_w$. Then, taking into account that A_w is the product of the family $(A_{w_i})_{i \in |w|}$, there exists, by the universal property of the product, a unique morphism $\langle P_i \rangle_{i \in |w|}$ from A_X to A_w such that, for every $i \in |w|$, the following diagram commutes

$$\begin{tikzcd} A_X \arrow[d, "\langle P_i \rangle_{i \in |w|}"'] \arrow[dr, "P_i"] & \\ A_w \arrow[r, "\mathrm{pr}_i"'] & A_{w_i} \end{tikzcd}$$

Then the structural operation F_σ is defined as:

$$F_\sigma \begin{cases} (A^{A_X})_w & \longrightarrow A_s^X \\ (P_i)_{i\in|w|} & \longmapsto F_\sigma^{\mathbf{A}} \circ \langle P_i \rangle_{i\in|w|} \end{cases}$$

where $F_\sigma^{\mathbf{A}}$ denotes the structural operation in $\mathbf{A}$ associated to σ.

For abbreviation we let X*-ary operations on* $\mathbf{A}$ stand for *many-sorted* X*-ary operations on* $\mathbf{A}$. The $\mathbf{\Sigma}$-algebra of the *many-sorted* X*-ary term operations on* $\mathbf{A}$, $\mathbf{Ter}_X(\mathbf{A})$, is the subalgebra of $\mathbf{O}_X(\mathbf{A})$ generated by

$$\mathcal{P}_X^A = (\mathcal{P}_{X,s}^A)_{s\in S} = (\{\, \mathrm{pr}_{X,s,x}^A \mid x \in X_s \,\})_{s\in S},$$

the subfamily of $\mathrm{O}_X(A) = A^{A_X}$, where, for every $s \in S$ and every $x \in X_s$, $\mathrm{pr}_{X,s,x}^A$ is the mapping from A_X to A_s which sends $a \in A_X$ to $a_s(x)$. For abbreviation we let X*-ary term operations on* $\mathbf{A}$ stand for *many-sorted* X*-ary term operations on* $\mathbf{A}$. We denote by $\mathrm{Tr}^{X,\mathbf{A}}$ the unique $\mathbf{\Sigma}$-homomorphism from $\mathbf{T}_{\mathbf{\Sigma}}(X)$ to $\mathbf{O}_X(\mathbf{A})$ such that $\mathrm{pr}_X^A = \mathrm{Tr}^{X,\mathbf{A}} \circ \eta_X$, where pr_X^A is the S-sorted mapping $(\mathrm{pr}_{X,s}^A)_{s\in S}$ from X to $\mathrm{O}_X(A)$ whose s-th coordinate, for each $s \in S$, is $\mathrm{pr}_{X,s}^A = (\mathrm{pr}_{X,s,x}^A)_{x\in X_s}$. For abbreviation, we let $P^{\mathbf{A}}$ stand for the image of P under $\mathrm{Tr}_s^{X,\mathbf{A}}$, and we call the mapping $P^{\mathbf{A}}$ from A_X to A_s, the *term operation on* $\mathbf{A}$ *determined by* P, or the *term realization of* P *on* $\mathbf{A}$ (also called by Computer Scientist the *evaluation* of P *on* $\mathbf{A}$). For simplicity of notation, we continue to write $\mathrm{Tr}^{X,\mathbf{A}}$ for the co-restriction of the $\mathbf{\Sigma}$-homomorphism $\mathrm{Tr}^{X,\mathbf{A}}\colon \mathbf{T}_{\mathbf{\Sigma}}(X) \longrightarrow \mathbf{O}_X(\mathbf{A})$ to the subalgebra $\mathbf{Ter}_X(\mathbf{A})$ of $\mathbf{O}_X(\mathbf{A})$.

We recall that, for $\eta_X\colon X \longrightarrow \mathrm{T}_{\mathbf{\Sigma}}(X)$, the insertion of the generators X into $\mathrm{T}_{\mathbf{\Sigma}}(X)$, $\mathrm{Tr}^{X,\mathbf{A}}[\eta_X[X]]$ is also $\mathbf{Ter}_X(\mathbf{A})$.

Remark. What we have called *term operations on* $\mathbf{A}$ are also known, for those following the terminology in Grätzer [17], pp. 37–45, and Jónsson [20], pp. 83–87, as *polynomial operations of* $\mathbf{A}$, and, for those following that one in Cohn [7], pp. 145–149, as *derived operators of* $\mathbf{A}$.

Following this we state the fundamental facts about term operations of different arities on the same algebra. These facts are, actually, the categorical counterpart and the generalization to the many-sorted case of some of those stated by Schmidt in [29], pp. 107–109.

Proposition 2.11. *Let* $\mathbf{A}$ *be a* $\mathbf{\Sigma}$*-algebra and* $f\colon X \longrightarrow Y$ *an* S*-sorted mapping. Then there exists a unique* $\mathbf{\Sigma}$*-homomorphism* $\mathrm{Ter}_f(\mathbf{A})$ *from* $\mathbf{Ter}_X(\mathbf{A})$ *to* $\mathbf{Ter}_Y(\mathbf{A})$ *such that* $\mathrm{Tr}^{Y,\mathbf{A}} \circ f^{@} = \mathrm{Ter}_f(\mathbf{A}) \circ \mathrm{Tr}^{X,\mathbf{A}}$*. Besides, for every* S*-sorted set* X*, we have that* $\mathrm{Ter}_{\mathrm{id}_X}(\mathbf{A}) = \mathrm{id}_{\mathbf{Ter}_X(\mathbf{A})}$*, and, if* $g\colon Y \longrightarrow Z$ *is another* S*-sorted mappings, then* $\mathrm{Ter}_{g\circ f}(\mathbf{A}) = \mathrm{Ter}_g(\mathbf{A}) \circ \mathrm{Ter}_f(\mathbf{A})$*.*

What we want to prove now is the compatibility between the translation of terms and their realization as term operations on the algebras. But for this it will be shown to be useful to take into account the following auxiliary functors and natural transformation.

Definition 2.12. For a mapping $\varphi\colon S \longrightarrow T$, an S-sorted set X, a T-sorted set Y, and an S-sorted mapping $f\colon X \longrightarrow Y_\varphi$, we have the following functors and natural transformation: (1) $\mathrm{H}(Y,\cdot)$ is the covariant hom-functor from $\mathbf{Set}^T$ to $\mathbf{Set}$, which, we recall, sends a T-sorted set A to the set $\mathrm{H}(Y,\cdot)(A) = A_Y$, and a T-sorted mapping u from A to B to the mapping $\mathrm{H}(Y,\cdot)(u)$ from A_Y to B_Y which assigns to a T-sorted mapping t from Y to A the mapping $u \circ t$ from Y to B, (2) $\mathrm{H}(X,\cdot) \circ \Delta_\varphi$ is the functor from $\mathbf{Set}^T$ to $\mathbf{Set}$ which sends a T-sorted set A to the set $(A_\varphi)_X$, and a T-sorted mapping u from A to B to the mapping $\mathrm{H}(X,\cdot)(u_\varphi)$ from $(A_\varphi)_X$ to $(B_\varphi)_X$ which assigns to an S-sorted mapping ℓ from X to A_φ the mapping $u_\varphi \circ \ell$ from X to B_φ, and (3) $\vartheta^{\varphi,f}$ is the natural transformation from $\mathrm{H}(Y,\cdot)$ to $\mathrm{H}(X,\cdot) \circ \Delta_\varphi$ which sends a T-sorted set A to the mapping $\vartheta^{\varphi,f}_A$ from A_Y to $(A_\varphi)_X$ which assigns to a morphism t in A_Y the morphism $t_\varphi \circ f$ in $(A_\varphi)_X$. Therefore, for a T-sorted set A, we have the S-sorted mapping $\Upsilon^{\varphi,f}_A$ from $\mathrm{O}_X(A_\varphi) = A_\varphi^{(A_\varphi)_X}$ to $\mathrm{O}_Y(A)_\varphi = (A^{A_Y})_\varphi = A_\varphi^{A_Y}$ which, for $s \in S$, sends $a\colon (A_\varphi)_X \longrightarrow A_{\varphi(s)}$ to $a \circ \vartheta^{\varphi,f}_A\colon A_Y \longrightarrow A_{\varphi(s)}$.

Proposition 2.13. *Let $(\varphi, d, f)\colon (S,\Sigma,X) \longrightarrow (T,\Lambda,Y)$ be a morphism in the category $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$. Then, for every $\mathbf{\Lambda}$-algebra $\mathbf{A}$ and every term $P \in \mathrm{T}_{\mathbf{\Sigma}}(X)_s$ for $\mathbf{\Sigma}$ of type (X,s), the mappings $P^{\mathbf{d}^*(\mathbf{A})} \circ \vartheta^{\varphi,f}_A$ and $f^{\mathbf{d}}_s(P)^{\mathbf{A}}$ from A_Y to $A_{\varphi(s)}$ are identical.*

Proof. Let $a \in A_Y$ be a T-sorted mapping from Y to A. Then the following diagram commutes

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "f"'] & \mathrm{T}_{\mathbf{\Sigma}}(X) \arrow[d, "f^{\mathbf{d}}"] \arrow[dd, bend left=60, "(a_\varphi \circ f)^\sharp"] \\
Y_\varphi \arrow[r, "(\eta_Y)_\varphi"] \arrow[dr, "a_\varphi"'] & \mathrm{T}_{\mathbf{\Lambda}}(Y)_\varphi \arrow[d, "(a^\sharp)_\varphi"] \\
 & A_\varphi
\end{tikzcd}$$

hence, for every $P \in \mathrm{T}_{\mathbf{\Sigma}}(X)_s$, we have that

$$\begin{aligned} f^{\mathbf{d}}_s(P)^{\mathbf{A}}(a) &= (a^{\sharp})_{\varphi(s)} \circ f^{\mathbf{d}}_s(P) \\ &= (a_\varphi \circ f)^{\sharp}_s(P) \\ &= P^{\mathbf{d}^*(\mathbf{A})}(a_\varphi \circ f) \\ &= P^{\mathbf{d}^*(\mathbf{A})} \circ \vartheta^{\varphi,f}_A(a). \end{aligned}$$

Therefore $f^{\mathbf{d}}_s(P)^{\mathbf{A}} = P^{\mathbf{d}^*(\mathbf{A})} \circ \vartheta^{\varphi,f}_A$, as asserted. ∎

We gather in the following corollary some useful consequences of the last proposition.

COROLLARY 2.14. *Let $(\varphi, d, f)\colon (S, \Sigma, X) \longrightarrow (T, \Lambda, Y)$ be a morphism in the category $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$, $\mathbf{A}$ a $\mathbf{\Lambda}$-algebra, and $P \in \mathrm{T}_{\mathbf{\Sigma}}(X)_s$ a term for $\mathbf{\Sigma}$ of type (X, s). Then we have that the following diagrams commute*

$$\begin{tikzcd} \mathbf{T}_{\mathbf{\Sigma}}(X) \arrow[r, "\mathrm{Tr}^{X,\mathbf{d}^*(\mathbf{A})}"] \arrow[d, "f^{\mathbf{d}}"'] & \mathbf{Ter}_X(\mathbf{d}^*(\mathbf{A})) \arrow[d, "\Upsilon^{\varphi,f}_A"] \\ \mathbf{T}_{\mathbf{\Lambda}}(Y)_\varphi \arrow[r, "\mathrm{Tr}^{Y,\mathbf{A}}_\varphi"'] & \mathbf{Ter}_Y(\mathbf{A})_\varphi \end{tikzcd} \qquad \begin{tikzcd} (A_\varphi)_X \arrow[r, "P^{\mathbf{d}^*(\mathbf{A})}"] & A_{\varphi(s)} \arrow[d, equal] \\ A_{\coprod_\varphi X} \arrow[u, "\theta^\varphi_{X,A}"] \arrow[r, "\eta^{\mathbf{d}}_{X,s}(P)^{\mathbf{A}}"'] & A_{\varphi(s)} \end{tikzcd}$$

where $\theta^\varphi_{X,A}$ is the component at (X, A) of the natural isomorphism θ^φ in Proposition 2.1.

PROOF. The left-hand diagram commutes because, for a morphism (φ, d, f) from (S, Σ, X) to (T, Λ, Y) and a $\mathbf{\Lambda}$-algebra $\mathbf{A}$, the S-sorted mapping $\Upsilon^{\varphi,f}_A$ from $\mathrm{O}_X(A_\varphi)$ to $\mathrm{O}_Y(A)_\varphi$ is actually a $\mathbf{\Sigma}$-homomorphism from $\mathbf{O}_X(\mathbf{d}^*(\mathbf{A}))$ to $\mathbf{O}_Y(\mathbf{A})_\varphi$ that restricts to $\mathbf{Ter}_X(\mathbf{d}^*(\mathbf{A}))$ and $\mathbf{Ter}_Y(\mathbf{A})_\varphi$.

The right-hand diagram commutes since, for the T-sorted set $\coprod_\varphi X$ and the S-sorted mapping η^φ_X from X to $(\coprod_\varphi X)_\varphi$, we have that $\vartheta^{\varphi,\eta^\varphi_X}_A = \theta^\varphi_{X,A}$. ∎

As it is well-known, for a signature $\mathbf{\Sigma}$, the conglomerate of terms for $\mathbf{\Sigma}$ is the set $\bigcup_{X \in \boldsymbol{\mathcal{U}}} \bigcup_{s \in S} \mathrm{T}_{\mathbf{\Sigma}}(X)_s$, but such an amorphous set is not adequate, because of its lack of structure, for some tasks, as e.g., to explain the invariant character of the realization of terms as term operations on algebras, under change of signature (or to state a Completeness Theorem for finitary many-sorted equational logic).

However, by conveniently generalizing the concept of term for a signature $\mathbf{\Sigma}$ (as explained immediately below), it is possible to equip, in a natural way, to the corresponding generalized terms for $\mathbf{\Sigma}$, taken as *morphisms*, with a category structure, that allows us to give a category-theoretic explanation of the relationship between terms and algebras. To this we add that the use of the generalized terms and related notions, such as, e.g., that of generalized equation (to be defined in the following section), has allowed us, in [5], to provide a purely category-theoretical proof of the Completeness Theorem for monads in categories of sorted sets.

Actually, we associate to every signature $\mathbf{\Sigma}$ the category $\mathbf{Kl}(\mathbb{T}_{\mathbf{\Sigma}})^{\mathrm{op}}$, of generalized terms for $\mathbf{\Sigma}$, that we denote, to shorten notation, by $\mathbf{Ter}(\mathbf{\Sigma})$, i.e., the dual of the Kleisli category for $\mathbb{T}_{\mathbf{\Sigma}} = (\mathrm{T}_{\mathbf{\Sigma}}, \eta, \mu)$, the standard monad derived from the adjunction $\mathbf{T}_{\mathbf{\Sigma}} \dashv \mathrm{G}_{\mathbf{\Sigma}}$ between the category $\mathbf{Alg}(\mathbf{\Sigma})$ and the category $\mathbf{Set}^S$, with $\mathrm{T}_{\mathbf{\Sigma}} = \mathrm{G}_{\mathbf{\Sigma}} \circ \mathbf{T}_{\mathbf{\Sigma}}$.

The construction of the category $\mathbf{Ter}(\mathbf{\Sigma})$ is a natural one. That this is so follows, essentially, from the fact that it has been obtained by applying a category-theoretic construction, concretely that of Kleisli (in [21]). However, to understand more plainly how the category $\mathbf{Ter}(\mathbf{\Sigma})$ is obtained, or, more precisely, from where the morphisms of $\mathbf{Ter}(\mathbf{\Sigma})$ arise, the following observation could be helpful. For a signature $\mathbf{\Sigma}$, an S-sorted set X, and a sort $s \in S$, an ordinary term $P \in \mathrm{T}_{\Sigma}(X)_s$ for $\mathbf{\Sigma}$ of type (X, s) is, essentially, an S-sorted mapping $P\colon \delta^s \longrightarrow \mathrm{T}_{\Sigma}(X)$ where, for $s \in S$, $\delta^s = (\delta^s_t)_{t \in S}$, the delta of Kronecker at s, is the S-sorted set such that $\delta^s_t = \varnothing$ if $s \neq t$ and $\delta^s_s = 1$. But the S-sorted mappings just mentioned do not constitute the morphisms of a category. Therefore, in order to obtain a category, it seems natural to replace the special S-sorted sets that are the deltas of Kronecker, as domains of morphisms, by arbitrary S-sorted sets, thus obtaining the generalized terms, that are the category-theoretic rendering of the ordinary terms, since they are now S-sorted mappings from an S-sorted set to the free $\mathbf{\Sigma}$-algebra on another S-sorted set, i.e., morphisms in a category $\mathbf{Ter}(\mathbf{\Sigma})$. This category-theoretic perspective about terms, in its turn, will allow us to obtain a functor $\mathrm{Tr}^{\mathbf{\Sigma}}$, of realization of terms as term operations, from $\mathbf{Alg}(\mathbf{\Sigma}) \times \mathbf{Ter}(\mathbf{\Sigma})$ to $\mathbf{Set}$, and therefore to define (in the next section) the validation of equations, understood as ordered pairs of coterminal terms in the corresponding generalized sense, in an algebra.

Since it will be fundamental in all that follows, we provide, for a signature $\mathbf{\Sigma}$, the full definition of the category $\mathbf{Ter}(\mathbf{\Sigma})$ and also the explicit definition of the procedure of realization of the terms for $\mathbf{\Sigma}$ as term operations on a given $\mathbf{\Sigma}$-algebra. Observe that we depart, in the definition of the category $\mathbf{Ter}(\mathbf{\Sigma})$, but only for this type of category, from the (non-Ehresmannian) tradition, in calling a category by the name of its morphisms.

Definition 2.15. Let $\mathbf{\Sigma}$ be a signature and $\mathbf{A}$ a $\mathbf{\Sigma}$-algebra. Then $\mathbf{Ter}(\mathbf{\Sigma})$, the category of *generalized terms for* $\mathbf{\Sigma}$, is the dual of $\mathbf{Kl}(\mathbb{T}_{\mathbf{\Sigma}})$: the objects are the elements of $\boldsymbol{\mathcal{U}}^S$; the morphisms from an S-sorted set X to another Y, which we call *generalized terms for* $\mathbf{\Sigma}$ *of type* (X, Y) (also called *substitutions* by Computer Scientist), or, for abbreviation, *terms of type* (X, Y), are the S-sorted mappings P from Y to $\mathrm{T}_{\mathbf{\Sigma}}(X)$; the composition, denoted in $\mathbf{Ter}(\mathbf{\Sigma})$ and $\mathbf{Kl}(\mathbb{T}_{\mathbf{\Sigma}})$ by $\diamond$, is the operation which sends $P\colon X \longrightarrow Y$ and $Q\colon Y \longrightarrow Z$ in $\mathbf{Ter}(\mathbf{\Sigma})$ to $Q \diamond P\colon X \longrightarrow Z$ in $\mathbf{Ter}(\mathbf{\Sigma})$, where $Q \diamond P$ is $\mu_X \circ P^{@} \circ Q$, with μ_X the value at X of the multiplication μ of the monad $\mathbb{T}_{\mathbf{\Sigma}}$ and $P^{@}$ the value of the functor $\mathbf{T}_{\mathbf{\Sigma}}$ at the S-sorted mapping $P\colon Y \longrightarrow \mathrm{T}_{\mathbf{\Sigma}}(X)$; and the identities are the values of η, the unit of the monad $\mathbb{T}_{\mathbf{\Sigma}}$, at the S-sorted sets. If $P\colon X \longrightarrow Y$ is a term for $\mathbf{\Sigma}$ of type (X, Y), then $P^{\mathbf{A}}$, the *term operation on* $\mathbf{A}$ *determined by* P, or the *term realization of* P *on* $\mathbf{A}$ (also called by Computer Scientist the *evaluation* of P *on* $\mathbf{A}$), is the mapping from A_X to A_Y which assigns to a valuation f of the variables X in A the valuation $f^\sharp \circ P$ of the variables Y in A.

After associating to every signature $\mathbf{\Sigma}$ the category $\mathbf{Ter}(\mathbf{\Sigma})$ of generalized terms, we proceed to assign to every signature morphism $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ a corresponding functor $\mathbf{d}_\diamond$ from $\mathbf{Ter}(\mathbf{\Sigma})$ to $\mathbf{Ter}(\mathbf{\Lambda})$.

Proposition 2.16. *Let* $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ *be a signature morphism. Then there exists a functor* $\mathbf{d}_\diamond$ *from* $\mathbf{Ter}(\mathbf{\Sigma})$ *to* $\mathbf{Ter}(\mathbf{\Lambda})$. *Its object mapping assigns to each* S*-sorted set* X *the* T*-sorted set* $\mathbf{d}_\diamond(X) = \coprod_\varphi X$*; its morphism mapping assigns to each morphism* P *from* X *to* Y *in* $\mathbf{Ter}(\mathbf{\Sigma})$ *the morphism* $\mathbf{d}_\diamond(P) = (\theta^\varphi)^{-1}(\eta^{\mathbf{d}}_X \circ P)$ *from* $\coprod_\varphi X$ *to* $\coprod_\varphi Y$ *in* $\mathbf{Ter}(\mathbf{\Lambda})$, *where* $\eta^{\mathbf{d}}_X$ *is the* $\mathbf{\Sigma}$*-homomorphism from* $\mathbf{T}_{\mathbf{\Sigma}}(X)$ *to* $\mathbf{T}_{\mathbf{\Lambda}}(\coprod_\varphi X)_\varphi$ *that extends the* S*-sorted mapping* $(\eta_{\coprod_\varphi X})_\varphi \circ \eta^\varphi_X$ *from* X *to* $\mathrm{T}_{\mathbf{\Lambda}}(\coprod_\varphi X)_\varphi$.

Proof. We restrict ourselves to prove that $\mathbf{d}_\diamond$ preserves compositions, since the proof that $\mathbf{d}_\diamond$ preserves identities is straightforward and for this reason the details are left to the reader. Let $P\colon X \longrightarrow Y$ and $Q\colon Y \longrightarrow Z$ be morphisms in $\mathbf{Ter}(\mathbf{\Sigma})$. Then we have the following equations:

$$
\begin{aligned}
\mathbf{d}_\diamond(Q \diamond P) &= (\theta^\varphi)^{-1}(\eta^{\mathbf{d}}_X \circ P^\sharp \circ Q) \\
&= (\theta^\varphi)^{-1}(\eta^{\mathbf{d}}_X) \circ \textstyle\coprod_\varphi P^\sharp \circ \coprod_\varphi Q, \\
\mathbf{d}_\diamond(Q) \diamond \mathbf{d}_\diamond(P) &= \mathbf{d}_\diamond(P)^\sharp \circ \mathbf{d}_\diamond(Q) \\
&= \mathbf{d}_\diamond(P)^\sharp \circ (\theta^\varphi)^{-1}(\eta^{\mathbf{d}}_Y) \circ \textstyle\coprod_\varphi Q,
\end{aligned}
$$

therefore, to prove that $\mathbf{d}_\diamond(Q \diamond P) = \mathbf{d}_\diamond(Q) \diamond \mathbf{d}_\diamond(P)$ it is enough to verify the following equation

$$(\theta^\varphi)^{-1}(\eta^{\mathbf{d}}_X) \circ \textstyle\coprod_\varphi P^\sharp = \mathbf{d}_\diamond(P)^\sharp \circ (\theta^\varphi)^{-1}(\eta^{\mathbf{d}}_Y).$$

But for this, because of the commutativity of the following diagram

$$\begin{tikzcd}
\coprod_\varphi \mathrm{T}_{\mathbf{\Sigma}}(Y) \arrow[r, "\coprod_\varphi \eta^{\mathbf{d}}_Y"] \arrow[d, "\coprod_\varphi P^\sharp"'] \arrow[rr, bend left, "(\theta^\varphi)^{-1}(\eta^{\mathbf{d}}_Y)"] & \coprod_\varphi \mathrm{T}_{\mathbf{\Lambda}}(\coprod_\varphi Y)_\varphi \arrow[r, "\varepsilon^\varphi_{\mathrm{T}_{\mathbf{\Lambda}}(\coprod_\varphi Y)}"] \arrow[d, "\coprod_\varphi \mathbf{d}_\diamond(P)^\sharp_\varphi"] & \mathrm{T}_{\mathbf{\Lambda}}(\coprod_\varphi Y) \arrow[d, "\mathbf{d}_\diamond(P)^\sharp"] \\
\coprod_\varphi \mathrm{T}_{\mathbf{\Lambda}}(X) \arrow[r, "\coprod_\varphi \eta^{\mathbf{d}}_X"'] \arrow[rr, bend right, "(\theta^\varphi)^{-1}(\eta^{\mathbf{d}}_X)"'] & \coprod_\varphi \mathrm{T}_{\mathbf{\Lambda}}(\coprod_\varphi X)_\varphi \arrow[r, "\varepsilon^\varphi_{\mathrm{T}_{\mathbf{\Lambda}}(\coprod_\varphi X)}"'] & \mathrm{T}_{\mathbf{\Lambda}}(\coprod_\varphi X)
\end{tikzcd}$$

it is enough to verify that the following equation holds

$$\eta^{\mathbf{d}}_X \circ P^\sharp = \mathbf{d}_\diamond(P)^\sharp_\varphi \circ \eta^{\mathbf{d}}_Y. \tag{1}$$

But equation (1) holds because the restriction of $\eta^{\mathbf{d}}_X \circ P^\sharp$ and $\mathbf{d}_\diamond(P)^\sharp_\varphi \circ \eta^{\mathbf{d}}_Y$ to the generating S-sorted set Y coincide, i.e., $\eta^{\mathbf{d}}_X \circ P^\sharp \circ \eta_Y = \mathbf{d}_\diamond(P)^\sharp_\varphi \circ \eta^{\mathbf{d}}_Y \circ \eta_Y$. In fact:

$$\eta^{\mathbf{d}}_X \circ P^\sharp \circ \eta_Y = \eta^{\mathbf{d}}_X \circ P,$$

$$\begin{aligned}
\mathbf{d}_\diamond(P)^\sharp_\varphi \circ \eta^{\mathbf{d}}_Y \circ \eta_Y &= \mathbf{d}_\diamond(P)^\sharp_\varphi \circ (\eta_{\coprod_\varphi Y})_\varphi \circ \eta^\varphi_Y \\
&= \mathbf{d}_\diamond(P)_\varphi \circ \eta^\varphi_Y \\
&= (\theta^\varphi)^{-1}(\eta^{\mathbf{d}}_X \circ P)_\varphi \circ \eta^\varphi_Y \\
&= (\eta^{\mathbf{d}}_X \circ P)^\sharp \circ \eta_Y \\
&= \eta^{\mathbf{d}}_X \circ P.
\end{aligned}$$

∎

Remark. For a generalized term P from X to Y, the generalized term $\mathbf{d}_\diamond(P)$ from $\coprod_\varphi X$ to $\coprod_\varphi Y$ can be defined alternative, but equivalently, as the composition of the morphisms in the following diagram

$$\coprod_{\varphi} Y \xrightarrow{\coprod_{\varphi} P} \coprod_{\varphi} \mathrm{T}_{\mathbf{\Sigma}}(X) \xrightarrow{\coprod_{\varphi} \eta^{\mathbf{d}}_{X}} \coprod_{\varphi} \mathrm{T}_{\mathbf{\Lambda}}(\coprod_{\varphi} X)_{\varphi} \xrightarrow{\varepsilon^{\varphi}_{\mathrm{T}_{\mathbf{\Lambda}}(\coprod_{\varphi} X)}} \mathrm{T}_{\mathbf{\Lambda}}(\coprod_{\varphi} X),$$

where, recalling that $\eta^{\mathbf{d}} = \mathrm{G}_{\mathbf{\Sigma}} * \eta^{\mathbf{d}}$ is the second natural transformation in Proposition 2.9 and ε^{φ} the counit of the adjunction $\coprod_{\varphi} \dashv \Delta_{\varphi}$, we have that

1. The T-sorted mapping $\coprod_{\varphi} \eta^{\mathbf{d}}_{X}$ is the component at X of the natural transformation $\coprod_{\varphi} * \eta^{\mathbf{d}}$ from $\coprod_{\varphi} \circ \mathrm{T}_{\mathbf{\Sigma}}$ to $\coprod_{\varphi} \circ \Delta_{\varphi} \circ \mathrm{T}_{\mathbf{\Lambda}} \circ \coprod_{\varphi}$, and
2. The T-sorted mapping $\varepsilon^{\varphi}_{\mathrm{T}_{\mathbf{\Lambda}}(\coprod_{\varphi} X)}$ is the component at X of the natural transformation $\varepsilon^{\varphi} * (\mathrm{T}_{\mathbf{\Lambda}} \circ \coprod_{\varphi} X)$ from $\coprod_{\varphi} \circ \Delta_{\varphi} \circ \mathrm{T}_{\mathbf{\Lambda}} \circ \coprod_{\varphi}$ to $\mathrm{T}_{\mathbf{\Lambda}} \circ \coprod_{\varphi} X$.

We state now for the generalized terms the homologous of the right-hand diagram in the first part of Corollary 2.14, i.e., the invariant character under signature change of the realization of generalized terms as term operations in algebras. We remark that from this fact we will obtain, in the third section, the invariance of the relation of satisfaction under signature change.

PROPOSITION 2.17. *Let $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ be a signature morphism. Then, for every $\mathbf{\Lambda}$-algebra $\mathbf{A}$ and every term P for $\mathbf{\Sigma}$ of type (X, Y), the mappings $P^{\mathbf{d}^{*}(\mathbf{A})} \circ \theta^{\varphi}_{X,A}$ and $\theta^{\varphi}_{Y,A} \circ \mathbf{d}_{\diamond}(P)^{\mathbf{A}}$ from $A_{\coprod_{\varphi} X}$ to $(A_{\varphi})_{Y}$ are identical.*

PROOF. Because the S-sorted set Y is isomorphic to $\coprod_{s \in S, y \in Y_s} \delta^{s}$ and the functor $\coprod_{\varphi}$ preserves colimits, since it has Δ_{φ} as a right adjoint, $\coprod_{\varphi} Y$ is isomorphic to $\coprod_{s \in S,\, y \in Y_s} \delta^{\varphi(s)}$. But $\mathrm{Hom}(\coprod_{\varphi} Y, A)$ and $\prod_{s \in S, y \in Y_s} \mathrm{Hom}(\delta^{\varphi(s)}, A)$ are isomorphic, thus it is enough to prove the proposition for the S-sorted sets of the type δ^{s}, i.e., the deltas of Kronecker, and this follows directly from Corollary 2.14. ∎

Once defined the mappings that associate, respectively, to a signature the corresponding category of terms, and to a signature morphism the functor between the associated categories of terms, we state in the following proposition that both mappings are actually the components of a pseudo-functor from **Sig** to the 2-category **Cat**.

PROPOSITION 2.18. *There exists a pseudo-functor* Ter *from* **Sig** *to the* 2*-category* **Cat** *given by the following data*

1. *The object mapping of* Ter *is that which sends a signature $\mathbf{\Sigma}$ to the category* $\mathrm{Ter}(\mathbf{\Sigma}) = \mathbf{Ter}(\mathbf{\Sigma})$.
2. *The morphism mapping of* Ter *is that which sends a signature morphism $\mathbf{d}$ from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$ to the functor* $\mathrm{Ter}(\mathbf{d}) = \mathbf{d}_{\diamond}$ *from* $\mathbf{Ter}(\mathbf{\Sigma})$ *to* $\mathbf{Ter}(\mathbf{\Lambda})$.

3. *For every* $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ *and* $\mathbf{e}\colon \mathbf{\Lambda} \longrightarrow \mathbf{\Omega}$*, the natural isomorphism* $\gamma^{\mathbf{d},\mathbf{e}}$ *from* $\mathbf{e}_\diamond \circ \mathbf{d}_\diamond$ *to* $(\mathbf{e}\circ\mathbf{d})_\diamond$ *is that which is defined, for every* S*-sorted set* X*, as the isomorphism* $\gamma^{\mathbf{d},\mathbf{e}}_X\colon \coprod_\psi \coprod_\varphi X \longrightarrow \coprod_{\psi\circ\varphi} X$ *in* $\mathbf{Ter}(\mathbf{\Omega})$ *that corresponds to the* U*-sorted mapping*

$$\coprod_{\psi\circ\varphi} X \xrightarrow{(\gamma^{\varphi,\psi}_X)^{-1}} \coprod_\psi \coprod_\varphi X \xrightarrow{\eta_{\coprod_\psi \coprod_\varphi X}} \mathrm{T}_{\mathbf{\Omega}}(\coprod_\psi \coprod_\varphi X),$$

where $\gamma^{\varphi,\psi}_X$ *is the component at* X *of the natural isomorphism* $\gamma^{\varphi,\psi}$ *for the pseudo-functor* $\mathrm{MSet}^{\amalg}$*.*

4. *For every signature* $\mathbf{\Sigma}$*, the natural isomorphism* $\nu^{\mathbf{\Sigma}}$ *from* $\mathrm{Id}_{\mathbf{Ter}(\mathbf{\Sigma})}$ *to* $(\mathrm{id}_{\mathbf{\Sigma}})_\diamond$ *is that which is defined, for every* S*-sorted set* X*, as the isomorphism* $\nu^{\mathbf{\Sigma}}_X\colon X \longrightarrow \coprod_{\mathrm{id}_S} X$ *in* $\mathbf{Ter}(\mathbf{\Sigma})$ *that corresponds to the* S*-sorted mapping*

$$\coprod_{\mathrm{id}_S} X \xrightarrow{\nu^S_X} X \xrightarrow{\eta_X} \mathrm{T}_{\mathbf{\Omega}}(X),$$

where ν^S_X *is the component at* X *of the natural isomorphism* ν^S *for the pseudo-functor* $\mathrm{MSet}^{\amalg}$*.*

What we want to achieve now is to prove, on the one hand, that, for a signature $\mathbf{\Sigma}$, there is a functor $\mathrm{Tr}^{\mathbf{\Sigma}}$ from $\mathbf{Alg}(\mathbf{\Sigma}) \times \mathbf{Ter}(\mathbf{\Sigma})$ to $\mathbf{Set}$, that simultaneously formalizes the procedure of realization of terms (as term operations on algebras), and its naturalness (by taking into account the variation of the algebras through the homomorphisms between them), and, on the other hand, that, for a signature morphism $\mathbf{d}$ from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$, there is a natural isomorphism $\theta^{\mathbf{d}}$ from $\mathrm{Tr}^{\mathbf{\Lambda}} \circ (\mathrm{Id}_{\mathbf{Alg}(\mathbf{\Lambda})} \times \mathbf{d}_\diamond)$ to $\mathrm{Tr}^{\mathbf{\Sigma}} \circ (\mathbf{d}^* \times \mathrm{Id}_{\mathbf{Ter}(\mathbf{\Sigma})})$, that shows the invariant character of the procedure of realization of terms under signature change.

To accomplish the first stated goal we begin by proving the following lemma.

Lemma 2.19. *Let* $\mathbf{A}$ *be a* $\mathbf{\Sigma}$*-algebra,* P *a term of type* (X,Y)*, and* Q *a term of type* (Y,Z)*. Then we have that* $(Q\diamond P)^{\mathbf{A}} = Q^{\mathbf{A}} \circ P^{\mathbf{A}}$*. Besides, for* η_X*, the identity morphism at* X *in* $\mathbf{Ter}(\mathbf{\Sigma})$*,* $\eta^{\mathbf{A}}_X = \mathrm{id}_{A_X}$*.*

Proof. We restrict ourselves to prove the first part of the lemma because the proof of the second one is straightforward. Since $(Q\diamond P)^{\mathbf{A}}$ is the mapping from A_X to A_Z which sends an S-sorted mapping $u\colon X \longrightarrow A$ to the S-sorted mapping $u^\sharp \circ (Q\diamond P) = u^\sharp \circ \mu_X \circ P^{@} \circ Q\colon Z \longrightarrow A$, (where, we recall, μ_X

is the value at X of the multiplication μ of the monad $\mathbb{T}_{\mathbf{\Sigma}} = (\mathrm{T}_{\mathbf{\Sigma}}, \eta, \mu)$ and $P^{@}$ the value at the S-sorted mapping $P\colon Y \longrightarrow \mathrm{T}_{\mathbf{\Sigma}}(X)$ of the functor $\mathbf{T}_{\mathbf{\Sigma}}$), and $Q^{\mathbf{A}} \circ P^{\mathbf{A}}$ the mapping from A_X to A_Z which sends an S-sorted mapping $u\colon X \longrightarrow A$ to the S-sorted mapping $(u^\sharp \circ P)^\sharp \circ Q\colon Z \longrightarrow A$, to verify that $(Q \diamond P)^{\mathbf{A}} = Q^{\mathbf{A}} \circ P^{\mathbf{A}}$ it is enough to prove that the $\mathbf{\Sigma}$-homomorphisms $u^\sharp \circ \mu_X \circ P^{@}$ and $(u^\sharp \circ P)^\sharp$ from $\mathbf{T}_{\mathbf{\Sigma}}(Y)$ to $\mathbf{A}$ are identical. But this follows from the equation $u^\sharp \circ \mu_X \circ P^{@} \circ \eta_Y = (u^\sharp \circ P)^\sharp \circ \eta_Y$, that is a consequence of the laws for the monad $\mathbb{T}_{\mathbf{\Sigma}}$ and of the equation $P^{@} \circ \eta_Y = \eta_{\mathrm{T}_{\mathbf{\Sigma}}(X)} \circ P$, where η_Y is the canonical embedding of Y into $\mathrm{T}_{\mathbf{\Sigma}}(Y)$ and $\eta_{\mathrm{T}_{\mathbf{\Sigma}}(X)}$ the canonical embedding of $\mathrm{T}_{\mathbf{\Sigma}}(X)$ into $\mathrm{T}_{\mathbf{\Sigma}}(\mathrm{T}_{\mathbf{\Sigma}}(X))$. ■

This lemma has as an immediate consequence the following corollary.

Corollary 2.20. *Let $\mathbf{\Sigma}$ be a signature and $\mathbf{A}$ a $\mathbf{\Sigma}$-algebra. Then there exists a functor $\mathrm{Tr}^{\mathbf{\Sigma},\mathbf{A}}$ from $\mathbf{Ter}(\mathbf{\Sigma})$ to $\mathbf{Set}$ which sends an S-sorted set X to the set $\mathrm{Tr}^{\mathbf{\Sigma},\mathbf{A}}(X) = A_X$ and a term $P\colon X \longrightarrow Y$ to the mapping $\mathrm{Tr}^{\mathbf{\Sigma},\mathbf{A}}(P) = P^{\mathbf{A}}\colon A_X \longrightarrow A_Y$, i.e., the term operation on $\mathbf{A}$ determined by P.*

Therefore, from the definition of the object and morphism mappings of the functors of the type $\mathrm{Tr}^{\mathbf{\Sigma},\mathbf{A}}$, we see that they encapsulate the procedure of realization of terms. Moreover, from the fact that they preserve identities and compositions in $\mathbf{Ter}(\mathbf{\Sigma})$, we conclude that they formally represent the two basic intuitions about the behaviour of the procedure just named, i.e., that the realization of an identity term is an identity term operation, and that the realization of a composite of two terms is the composite of their respective realizations (in the same order).

Before stating the following lemma we recall that, for an S-sorted mapping f from an S-sorted set A into another B and an S-sorted set X, f_X is the value at X of the natural transformation $\mathrm{H}(\cdot, f)$ from the contravariant functor $\mathrm{H}(\cdot, A)$ to the contravariant functor $\mathrm{H}(\cdot, B)$, both from $(\mathbf{Set}^S)^{\mathrm{op}}$ to $\mathbf{Set}$.

Lemma 2.21. *Let f be a $\mathbf{\Sigma}$-homomorphism from $\mathbf{A}$ to $\mathbf{B}$ and P a term of type (X, Y) in $\mathbf{Ter}(\mathbf{\Sigma})$. Then the mappings $P^{\mathbf{B}} \circ f_X$ and $f_Y \circ P^{\mathbf{A}}$ from A_X to B_Y are identical, and we agree to denote it by f_P.*

Proof. Given an S-sorted mapping $u\colon X \longrightarrow A$, we have that $(f \circ u)^\sharp = f \circ u^\sharp$, by the universal property of the free $\mathbf{\Sigma}$-algebra on X and taking into account that f is a $\mathbf{\Sigma}$-homomorphism from $\mathbf{A}$ to $\mathbf{B}$. Therefore, since $P^{\mathbf{B}} \circ f_X(u) = (f \circ u)^\sharp \circ P$, and $f_Y \circ P^{\mathbf{A}}(u) = f \circ (u^\sharp \circ P)$, we have that $P^{\mathbf{B}} \circ f_X(u) = f_Y \circ P^{\mathbf{A}}(u)$. Thus $P^{\mathbf{B}} \circ f_X = f_Y \circ P^{\mathbf{A}}$. ■

This lemma has as an immediate consequence the following corollary.

COROLLARY 2.22. *Let* $\mathbf{\Sigma}$ *be a signature and* f *a* $\mathbf{\Sigma}$*-homomorphism from* $\mathbf{A}$ *to* $\mathbf{B}$. *Then there exists a natural transformation* $\mathrm{Tr}^{\mathbf{\Sigma},f}$ *from the functor* $\mathrm{Tr}^{\mathbf{\Sigma},\mathbf{A}}$ *to the functor* $\mathrm{Tr}^{\mathbf{\Sigma},\mathbf{B}}$*, as reflected in the diagram*

$$\begin{tikzcd} \mathbf{Ter}(\mathbf{\Sigma}) \arrow[rr, bend left, "\mathrm{Tr}^{\mathbf{\Sigma},\mathbf{A}}"] \arrow[rr, bend right, "\mathrm{Tr}^{\mathbf{\Sigma},\mathbf{B}}"'] \arrow[rr, phantom, "\Downarrow \mathrm{Tr}^{\mathbf{\Sigma},f}"] & & \mathbf{Set} \end{tikzcd}$$

which sends an S*-sorted set* X *to the mapping* $\mathrm{Tr}^{\mathbf{\Sigma},f}_X = f_X$ *from* A_X *to* B_X. *Besides, for* $\mathrm{id}_{\mathbf{A}}$*, the identity* $\mathbf{\Sigma}$*-homomorphism at* $\mathbf{A}$*, we have that* $\mathrm{Tr}^{\mathbf{\Sigma},\mathrm{id}_{\mathbf{A}}} = \mathrm{id}_{\mathrm{Tr}^{\mathbf{\Sigma},\mathbf{A}}}$*, and, if* $g\colon \mathbf{B} \longrightarrow \mathbf{C}$ *is another* $\mathbf{\Sigma}$*-homomorphism, then* $\mathrm{Tr}^{\mathbf{\Sigma},g\circ f} = \mathrm{Tr}^{\mathbf{\Sigma},g} \circ \mathrm{Tr}^{\mathbf{\Sigma},f}$.

Therefore, the naturalness of the procedure of realization of terms as term operations on the different algebras is embodied in the natural transformations of the type $\mathrm{Tr}^{\mathbf{\Sigma},f}$.

REMARK. By identifying the $\mathbf{\Sigma}$-homomorphisms with the $\mathbb{T}_{\mathbf{\Sigma}}$-homomorphisms, the just stated corollary can be interpreted as meaning that every $\mathbf{\Sigma}$-homomorphism f from $\mathbf{A}$ to $\mathbf{B}$ is a natural transformation from the functor $\mathrm{Tr}^{\mathbf{\Sigma},\mathbf{A}}$ to the functor $\mathrm{Tr}^{\mathbf{\Sigma},\mathbf{B}}$, both from $\mathbf{Ter}(\mathbf{\Sigma}) = \mathbf{Kl}(\mathbb{T}_{\mathbf{\Sigma}})^{\mathrm{op}}$ to $\mathbf{Set}$. Actually, each homomorphism $(\mathbf{d}, f)$ from an algebra $(\mathbf{\Sigma}, \mathbf{A})$ into another $(\mathbf{\Lambda}, \mathbf{B})$ is identifiable to a morphism (in the category $(\mathbf{Cat})_{//\mathbf{Set}}$, see [18], p. (sub) 186) from the object $(\mathbf{Ter}(\mathbf{\Sigma}), \mathrm{Tr}^{\mathbf{\Sigma},\mathbf{A}})$ over $\mathbf{Set}$ to the object $(\mathbf{Ter}(\mathbf{\Lambda}), \mathrm{Tr}^{\mathbf{\Lambda},\mathbf{B}})$ over $\mathbf{Set}$, concretely, to the morphism given by the pair $(\mathbf{d}_\diamond, (\theta^{\varphi}_{\cdot,B})^{-1} \circ \mathrm{H}(\cdot, f))$, where $\mathrm{H}(\cdot, f)$ is the natural transformation from the contravariant hom-functor $\mathrm{H}(\cdot, A)$ to the contravariant hom-functor $\mathrm{H}(\cdot, B_\varphi)$, and $(\theta^{\varphi}_{\cdot,B})^{-1}$ the natural isomorphism from $\mathrm{H}(\cdot, B_\varphi)$ to $\mathrm{H}(\coprod_\varphi(\cdot), B)$. Observe that the naturalness of $(\theta^{\varphi}_{\cdot,B})^{-1} \circ \mathrm{H}(\cdot, f)$ means that, for every term P for $\mathbf{\Sigma}$ of type (X, Y), the mappings $(\theta^{\varphi}_{Y,B})^{-1} \circ \mathrm{H}(Y, f) \circ P^{\mathbf{A}}$ and $\mathbf{d}_\diamond(P)^{\mathbf{B}} \circ (\theta^{\varphi}_{X,B})^{-1} \circ \mathrm{H}(X, f)$ from A_X to $B_{\coprod_\varphi Y}$ are identical.

From the identification of the homomorphisms between algebras in the category $\mathbf{Alg}$ to some convenient morphisms between the associated objects over $\mathbf{Set}$, we can conclude, e.g., that the concept of homomorphism as defined by Bénabou in [2] (that does not allow the variation of the signature and therefore it works between algebras of the same signature (see [2], p. (sub) 16, last paragraph)), corresponds itself, for a signature $\mathbf{\Sigma}$

and a $\mathbf{\Sigma}$-homomorphism f from $\mathbf{A}$ to $\mathbf{B}$, to the (very special) case in which $(\mathbf{d}_\diamond, (\theta^{\varphi}_{\cdot,B})^{-1} \circ \mathrm{H}(\cdot, f))$ is precisely

$$(\mathbf{d}_\diamond, (\theta^{\varphi}_{\cdot,B})^{-1} \circ \mathrm{H}(\cdot, f)) = (\mathrm{Id}_{\mathbf{Ter}(\mathbf{\Sigma})}, \mathrm{H}(\cdot, f)),$$

i.e., definitely, it corresponds to the natural transformation $\mathrm{Tr}^{\mathbf{\Sigma},f}$ from the functor $\mathrm{Tr}^{\mathbf{\Sigma},\mathbf{A}}$ to the functor $\mathrm{Tr}^{\mathbf{\Sigma},\mathbf{B}}$.

For a signature $\mathbf{\Sigma}$ the family of functors $(\mathrm{Tr}^{\mathbf{\Sigma},\mathbf{A}})_{\mathbf{A}\in\mathbf{Alg}(\mathbf{\Sigma})}$ together with the family of natural transformations $(\mathrm{Tr}^{\mathbf{\Sigma},f})_{f\in\mathrm{Mor}(\mathbf{Alg}(\mathbf{\Sigma}))}$ are the object and morphism mappings, respectively, of a functor $\mathrm{Tr}^{\mathbf{\Sigma},(\cdot)}$ from the category $\mathbf{Alg}(\mathbf{\Sigma})$ to the exponential category $\mathbf{Set}^{\mathbf{Ter}(\mathbf{\Sigma})}$, and the functor $\mathrm{Tr}^{\mathbf{\Sigma},(\cdot)}$ will allow us to prove in the following proposition, that there exists a functor $\mathrm{Tr}^{\mathbf{\Sigma}}$ from $\mathbf{Alg}(\mathbf{\Sigma}) \times \mathbf{Ter}(\mathbf{\Sigma})$ to $\mathbf{Set}$ that formalizes the realization of terms as term operations on algebras, but taking into account the variation of the algebras through the homomorphisms between them.

Proposition 2.23. *Let $\mathbf{\Sigma}$ be a signature. Then there exists a functor $\mathrm{Tr}^{\mathbf{\Sigma}}$ from $\mathbf{Alg}(\mathbf{\Sigma}) \times \mathbf{Ter}(\mathbf{\Sigma})$ to $\mathbf{Set}$. Its object mapping assigns to each pair $(\mathbf{A}, X)$, formed by a $\mathbf{\Sigma}$-algebra $\mathbf{A}$ and an S-sorted set X, the set $\mathrm{Tr}^{\mathbf{\Sigma}}(\mathbf{A}, X) = \mathrm{Tr}^{\mathbf{\Sigma},\mathbf{A}}(X) = A_X$ of the S-sorted mappings from X to the underlying S-sorted set A of $\mathbf{A}$; its morphism mapping assigns to each arrow (f, P) from $(\mathbf{A}, X)$ to $(\mathbf{B}, Y)$ in $\mathbf{Alg}(\mathbf{\Sigma}) \times \mathbf{Ter}(\mathbf{\Sigma})$, the mapping $\mathrm{Tr}^{\mathbf{\Sigma}}(f, P) = f_P$ from A_X to B_Y, which is precisely*

$$\mathrm{Tr}^{\mathbf{\Sigma},\mathbf{B}}(P) \circ \mathrm{Tr}^{\mathbf{\Sigma},f}_X = \mathrm{Tr}^{\mathbf{\Sigma},f}_Y \circ \mathrm{Tr}^{\mathbf{\Sigma},\mathbf{A}}(P).$$

Proof. It follows, essentially, after Lemma 2.19. ■

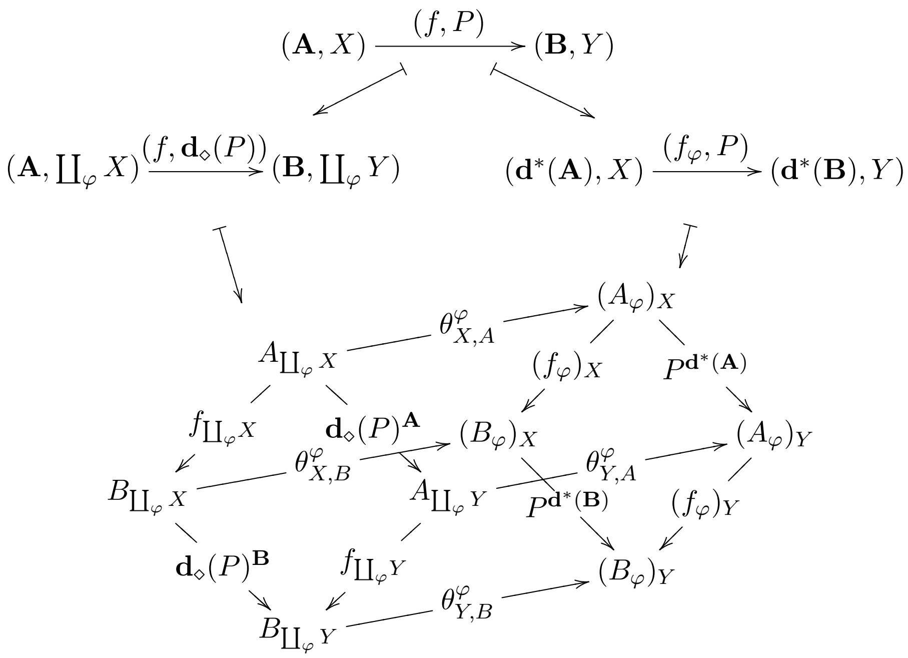
To accomplish the earlier second stated goal, i.e., to show the invariant character of the procedure of realization of terms under signature change, we prove in the following proposition, for a morphism $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$, the existence of a natural isomorphism between two functors from $\mathbf{Alg}(\mathbf{\Lambda}) \times \mathbf{Ter}(\mathbf{\Sigma})$ to $\mathbf{Set}$, constructed from the functors $\mathrm{Tr}^{\mathbf{\Lambda}}$, $\mathrm{Tr}^{\mathbf{\Sigma}}$, $\mathbf{d}_\diamond$ and $\mathbf{d}^*$.

Proposition 2.24. *Let $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ be a signature morphism. Then the family $(\theta^{\mathbf{d}}_{\mathbf{A},X})_{(\mathbf{A},X)\in\mathbf{Alg}(\mathbf{\Lambda})\times\mathbf{Ter}(\mathbf{\Sigma})}$, written $\theta^{\mathbf{d}}$ for brevity, where, for each pair $(\mathbf{A}, X)$ in $\mathbf{Alg}(\mathbf{\Lambda}) \times \mathbf{Ter}(\mathbf{\Sigma})$, $\theta^{\mathbf{d}}_{\mathbf{A},X}$ is $\theta^{\varphi}_{X,A}$, i.e., the component at (X, A) of the natural isomorphism of $\coprod_\varphi \dashv \Delta_\varphi$, is a natural isomorphism from the functor $\mathrm{Tr}^{\mathbf{\Lambda}} \circ (\mathrm{Id} \times \mathbf{d}_\diamond)$ to the functor $\mathrm{Tr}^{\mathbf{\Sigma}} \circ (\mathbf{d}^* \times \mathrm{Id})$, both from the category*

$\mathbf{Alg}(\mathbf{\Lambda}) \times \mathbf{Ter}(\mathbf{\Sigma})$ *to the category* $\mathbf{Set}$*, as shown in the following diagram*

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbf{Alg}(\mathbf{\Lambda}) \times \mathbf{Ter}(\mathbf{\Sigma}) \arrow[r, "\mathbf{d}^* \times \mathrm{Id}"] \arrow[d, "\mathrm{Id} \times \mathbf{d}_\diamond"'] \arrow[dr, phantom, "\theta^{\mathbf{d}} \nearrow"] & \mathbf{Alg}(\mathbf{\Sigma}) \times \mathbf{Ter}(\mathbf{\Sigma}) \arrow[d, "\mathrm{Tr}^{\mathbf{\Sigma}}"] \\
\mathbf{Alg}(\mathbf{\Lambda}) \times \mathbf{Ter}(\mathbf{\Lambda}) \arrow[r, "\mathrm{Tr}^{\mathbf{\Lambda}}"'] & \mathbf{Set}
\end{tikzcd}$$

PROOF. Let $(f, P)\colon (\mathbf{A}, X) \longrightarrow (\mathbf{B}, Y)$ be a morphism in $\mathbf{Alg}(\mathbf{\Lambda}) \times \mathbf{Ter}(\mathbf{\Sigma})$. Then we have the following situation



But the bottom diagram in the above figure commutes, because of Proposition 2.17, the naturalness of θ^φ, and the fact that f is a $\mathbf{\Lambda}$-homomorphism. Therefore the mappings $\theta^\varphi_{Y,B} \circ f_{\mathbf{d}_\diamond(P)}$ and $(f_\varphi)_P \circ \theta^\varphi_{X,A}$ from $A_{\coprod_\varphi X}$ to $(B_\varphi)_Y$ are identical. From this it follows that the family $\theta^{\mathbf{d}}$ is a natural isomorphism from $\mathrm{Tr}^{\mathbf{\Lambda}} \circ (\mathrm{Id} \times \mathbf{d}_\diamond)$ to $\mathrm{Tr}^{\mathbf{\Sigma}} \circ (\mathbf{d}^* \times \mathrm{Id})$. ■

Our next goal is to construct the many-sorted term institution by combining adequately the above components, i.e., the contravariant functor Alg from **Sig** to **Cat**, the pseudo-functor Ter from **Sig** to **Cat**, the family of functors $\mathrm{Tr} = (\mathrm{Tr}^{\mathbf{\Sigma}})_{\mathbf{\Sigma}\in\mathbf{Sig}}$, and the family of natural isomorphisms $\theta = (\theta^{\mathbf{d}})_{\mathbf{d}\in\mathrm{Mor}(\mathbf{Sig})}$.

To attain the just stated goal we need to recall beforehand some auxiliary concepts. In particular, we proceed to define next, among others, the concept of pseudo-extranatural transformation in 2-categories and for pseudo-functors. This generality is necessary as will be pointed out afterwards.

Definition 2.25. Let $\mathbf{C}$ and $\mathbf{D}$ be two 2-categories, $F, G\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \longrightarrow \mathbf{D}$ two 2-functors, and (α, β) a pair such that

1. For every 0-cell c in $\mathbf{C}$, $\alpha_c\colon F(c, c) \longrightarrow G(c, c)$ is a 1-cell in $\mathbf{D}$.
2. For every 1-cell $f\colon c \longrightarrow c'$ in $\mathbf{C}$, β_f is a 2-cell in $\mathbf{D}$ from the composite $G(1, f) \circ \alpha_c \circ F(f, 1)$ to the composite $G(f, 1) \circ \alpha_{c'} \circ F(1, f)$.

Then we say that (α, β) is a

1. *Lax-dinatural transformation* from F to G if, for every 2-cell $\xi\colon f \Rightarrow g$ in $\mathbf{C}$, we have that
$$\beta_g \circ (G(1, \xi) * \alpha_c * F(\xi, 1)) = (G(\xi, 1) * \alpha_{c'} * F(1, \xi)) \circ \beta_f.$$
2. *Pseudo-dinatural transformation* from F to G if it is a lax-dinatural transformation and, for every $f\colon c \longrightarrow c'$ in $\mathbf{C}$, β_f is an isomorphism.
3. 2-*dinatural transformation* from F to G if it is a lax-dinatural transformation and, for every $f\colon c \longrightarrow c'$ in $\mathbf{C}$, β_f is an identity.

The dinatural transformations when F and G are pseudo-functors will also be relevant for us. In this case it is necessary to impose additional conditions of compatibility with the natural isomorphisms of the pseudo-functors. The definition is as follows.

Definition 2.26. Let $\mathbf{C}$, $\mathbf{D}$ be two 2-categories, (F, γ^F, ν^F), (G, γ^G, ν^G) two pseudo-functors from $\mathbf{C}^{\mathrm{op}} \times \mathbf{C}$ to $\mathbf{D}$, and (α, β) a pair such that

1. For every 0-cell c in $\mathbf{C}$, $\alpha_c\colon F(c, c) \longrightarrow G(c, c)$ is a 1-cell in $\mathbf{D}$.
2. For every 1-cell $f\colon c \longrightarrow c'$ in $\mathbf{C}$, β_f is a 2-cell in $\mathbf{D}$ from the composite $G(1, f) \circ \alpha_c \circ F(f, 1)$ to the composite $G(f, 1) \circ \alpha_{c'} \circ F(1, f)$.

Then we say that (α, β) is a *lax-dinatural transformation* from (F, γ^F, ν^F) to (G, γ^G, ν^G) if it satisfies the following compatibility conditions:

1. For every 2-cell $\xi \colon f \Rightarrow g$ in $\mathbf{C}$, we have that
$$\beta_g \circ (G(1, \xi) * \alpha_c * F(\xi, 1)) = (G(\xi, 1) * \alpha_{c'} * F(1, \xi)) \circ \beta_f.$$
2. For every pair of 1-cells $f \colon c \longrightarrow c'$, $g \colon c' \longrightarrow c''$ in $\mathbf{C}$, we have that
$$\gamma^F_{(1,f),(1,g)} \circ \big(G(f,1) * \beta_g * F(1,f)\big) \circ \big(G(1,g) * \beta_f * F(g,1)\big) \\ = \beta_{g \circ f} \circ \big(\gamma^G_{(1,f),(1,g)} * \alpha_c * \gamma^F_{(g,1),(f,1)}\big).$$
3. For every object c in $\mathbf{C}$, we have that
$$\alpha_c * \nu^F_{(c,c)} = \nu^G_{(c,c)} * \alpha_c.$$

If the pseudo-functor G is independent of both variables, then we say that the above transformations are *lax-extranatural*, *pseudo-extranatural* or *extranatural*, respectively. Then the compatibility with the 2-cells of $\mathbf{C}$ is equivalent to
$$\beta_g \circ (\alpha_c * F(\xi, 1)) = (\alpha_{c'} * F(1, \xi)) \circ \beta_f,$$
and the compatibility of the composition of 1-cells in $\mathbf{C}$ with the natural isomorphisms of F is equivalent to
$$\gamma^F_{(1,f),(1,g)} \circ (\beta_g * F(1,f)) \circ (\beta_f * F(g,1)) = \beta_{g \circ f} \circ (\alpha_c * \gamma^F_{(g,1),(f,1)}).$$

In the following proposition we define a pseudo-functor from the category $\mathbf{Sig}^{\mathrm{op}} \times \mathbf{Sig}$ to $\mathbf{Cat}$, and prove that there exists a pseudo-extranatural transformation from it to the functor from $\mathbf{Sig}^{\mathrm{op}} \times \mathbf{Sig}$ to $\mathbf{Cat}$ which picks $\mathbf{Set}$.

Proposition 2.27. *There exists a pseudo-functor* $\mathrm{Alg}(\cdot) \times \mathrm{Ter}(\cdot)$ *from the category* $\mathbf{Sig}^{\mathrm{op}} \times \mathbf{Sig}$ *to* $\mathbf{Cat}$*, obtained from the contravariant functor* Alg *and the pseudo-functor* Ter*, which sends a pair of signatures* $(\boldsymbol{\Sigma}, \boldsymbol{\Lambda})$ *to the category* $\mathbf{Alg}(\boldsymbol{\Sigma}) \times \mathbf{Ter}(\boldsymbol{\Lambda})$*, and a pair of signature morphisms* $(\mathbf{d}, \mathbf{e})$ *from* $(\boldsymbol{\Sigma}, \boldsymbol{\Lambda})$ *to* $(\boldsymbol{\Sigma}', \boldsymbol{\Lambda}')$ *in* $\mathbf{Sig}^{\mathrm{op}} \times \mathbf{Sig}$ *to the functor* $\mathbf{d}^* \times \mathbf{e}_\diamond$ *from* $\mathbf{Alg}(\boldsymbol{\Sigma}) \times \mathbf{Ter}(\boldsymbol{\Lambda})$ *to* $\mathbf{Alg}(\boldsymbol{\Sigma}') \times \mathbf{Ter}(\boldsymbol{\Lambda}')$*. Furthermore, the family of functors* $\mathrm{Tr} = (\mathrm{Tr}^{\boldsymbol{\Sigma}})_{\boldsymbol{\Sigma} \in \mathbf{Sig}}$*, together with the family* $\theta = (\theta^{\mathbf{d}})_{\mathbf{d} \in \mathrm{Mor}(\mathbf{Sig})}$*, where* $\theta^{\mathbf{d}}$ *is the natural isomorphism of Proposition 2.24, is a pseudo-extranatural transformation from the pseudo-functor* $\mathrm{Alg}(\cdot) \times \mathrm{Ter}(\cdot)$ *to the functor* $\mathrm{K}_{\mathbf{Set}}$*, which picks* $\mathbf{Set}$*, both from* $\mathbf{Sig}^{\mathrm{op}} \times \mathbf{Sig}$ *to* $\mathbf{Cat}$*.*

Proof. Because the 2-category structure of **Sig** is, in this case, trivial, we need only prove the compatibility with the natural isomorphisms of the pseudo-functor $\mathrm{Alg}(\cdot) \times \mathrm{Ter}(\cdot)$.

We restrict our attention to prove the compatibility of the composition of 1-cells in **Sig** with the natural isomorphisms of $\mathrm{Alg}(\cdot) \times \mathrm{Ter}(\cdot)$. But for this, it is enough to verify that, for every $f\colon \mathbf{A} \longrightarrow \mathbf{B}$ in $\mathbf{Alg}(\boldsymbol{\Omega})$ and $P\colon X \longrightarrow Y$ in $\mathbf{Ter}(\boldsymbol{\Sigma})$, the following diagram commutes

$$
\begin{tikzcd}
 & (A_{\varphi})_{\coprod_{\varphi} X} \arrow[rr, "\theta^{\varphi}_{X,A_{\psi}}"] \arrow[dd, "\mathbf{d}_{\diamond}(P)^{\mathbf{e}^{*}(\mathbf{A})}"] & & (A_{\psi_{\varphi}})_{X} \arrow[dd, "P^{\mathbf{d}^{*}(\mathbf{e}^{*}(\mathbf{A}))}"] \\
A_{\coprod_{\psi}\coprod_{\varphi} X} \arrow[ur, "\theta^{\psi}_{\coprod_{\varphi} X,A}"] \arrow[rr, "(\gamma^{\mathbf{d},\mathbf{e}}_{X})^{\mathbf{A}}"] \arrow[dd, "\mathbf{e}_{\diamond} \circ \mathbf{d}_{\diamond}(P)^{\mathbf{A}}"] & & A_{\coprod_{\psi\circ\varphi} X} \arrow[ur, "\theta^{\psi\circ\varphi}_{X,A}"] \arrow[dd, "\mathbf{e}_{\diamond}(\mathbf{d}_{\diamond}(P))^{\mathbf{A}}"] & \\
 & (A_{\varphi})_{\coprod_{\varphi} Y} \arrow[rr, "\theta^{\varphi}_{Y,A_{\psi}}"] \arrow[dd, "(f_{\varphi})_{\coprod_{\varphi} Y}"] & & (A_{\psi_{\varphi}})_{Y} \arrow[dd, "(f_{\psi_{\varphi}})_{Y}"] \\
A_{\coprod_{\psi}\coprod_{\varphi} Y} \arrow[ur, "\theta^{\psi}_{\coprod_{\varphi} Y,A}"] \arrow[rr, "(\gamma^{\mathbf{d},\mathbf{e}}_{Y})^{\mathbf{A}}"] \arrow[dd, "f_{\coprod_{\psi}\coprod_{\varphi} Y}"] & & A_{\coprod_{\psi\circ\varphi} Y} \arrow[ur, "\theta^{\psi\circ\varphi}_{Y,A}"] \arrow[dd, "f_{\coprod_{\psi\circ\varphi} Y}"] & \\
 & (B_{\psi})_{\coprod_{\varphi} Y} \arrow[rr, "\theta^{\varphi}_{Y,B_{\psi}}"] & & (B_{\psi_{\varphi}})_{Y} \\
B_{\coprod_{\psi}\coprod_{\varphi} Y} \arrow[ur, "\theta^{\psi}_{\coprod_{\varphi} Y,B}"] \arrow[rr, "(\gamma^{\mathbf{d},\mathbf{e}}_{Y})^{\mathbf{B}}"] & & B_{\coprod_{\psi\circ\varphi} Y} \arrow[ur, "\theta^{\psi\circ\varphi}_{Y,B}"] &
\end{tikzcd}
$$

And this is so in consequence of the definitions of the involved entities. ∎

To actually realize the announced reformulation of the proposition just stated we should begin by defining a concept of institution that generalizes, even more, that one defined by Goguen and Burstall in [15]. This generalization is founded, ultimately, on the fact that the compatibility of generalized many-sorted terms and many-sorted algebras with respect to transformations between many-sorted signatures is also valid when generalized many-sorted terms and many-sorted algebras (of different many-sorted signatures) are equipped with natural category structures.

Definition 2.28. Let $\mathbf{C}$ be a category. Then a 2-*institution on* $\mathbf{C}$ is a quadruple $(\mathbf{Sig}, \mathrm{Mod}, \mathrm{Sen}, (\alpha, \beta))$, where

1. $\mathbf{Sig}$ is a 2-category.
2. $\mathrm{Mod}\colon \mathbf{Sig}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ a pseudo-functor.
3. $\mathrm{Sen}\colon \mathbf{Sig} \longrightarrow \mathbf{Cat}$ a pseudo-functor.
4. $(\alpha, \beta)\colon\ \mathrm{Mod}(\cdot) \times \mathrm{Sen}(\cdot) \longrightarrow \mathrm{K}_{\mathbf{C}}$ a pseudo-extranatural transformation.

If $\mathbf{Sig}$ is an ordinary category, instead of a 2-category, then we will speak of an *institution on* $\mathbf{C}$.

Remark. The concept of 2-institution is defined relative to a category, i.e., it has meaning for a 0-cell $\mathbf{C}$ of the 2-category $\mathbf{Cat} = 1-\mathbf{Cat}$, of categories, functors, and natural transformations between functors. Therefore, if it were necessary for some application, the concept of 3-institution ought to be defined relative to a 0-cell $\mathbf{C}$ of the 3-category $2-\mathbf{Cat}$, of 2-categories, 2-functors, 2-natural transformations and modifications between transformations, and so forth.

Remark. Actually, 2-institutions and institutions on a category, if they are understood as pseudo-extranatural transformations, go beyond both the classical conception of semantical truth defined (mathematically for the first time, through a recursive definition of satisfaction of a formula in an arbitrary relational system by a valuation of the variables in the system) by Tarski and Vaught in [31], p. 85, and the latest conception of institution in Goguen and Burstall [15], p. 327.

From the above it follows, immediately, the following corollary.

Corollary 2.29. *The quadruple* $\mathfrak{Tm} = (\mathbf{Sig}, \mathrm{Alg}, \mathrm{Ter}, (\mathrm{Tr}, \theta))$ *is an institution on the category* $\mathbf{Set}$*, the so-called* many-sorted term institution*, or, to abbreviate, the* term institution*.*

We close this section by pointing out that the institution $\mathfrak{Tm}$ can be qualified of basic, or fundamental, among others, by the following reasons: (1) it embodies, in a coherent way, algebras, terms, and the natural procedure of realization of terms as term operations in algebras, and (2) the many-sorted equational institution and the many-sorted specification institution (both of them defined in the following section), i.e., the core of universal algebra, are built on it.

On the other hand, let us notice that in this article, as we have said in the introduction, no attempt has been made to provide examples of 2-categories on given categories. The interested reader can easily obtain them from those results contained in [6] which have to do with the polyderivors and the transformations between polyderivors.

3. Many-sorted specifications and morphisms.

In this section we begin by defining, for a signature $\mathbf{\Sigma}$, the concept of $\mathbf{\Sigma}$-equation, but for the generalized terms defined in the preceding section, the binary relation of satisfaction between $\mathbf{\Sigma}$-algebras and $\mathbf{\Sigma}$-equations, and the semantical consequence operators $\mathrm{Cn}_{\mathbf{\Sigma}}$. Then, after extending the translation of generalized terms to generalized equations, we prove the corresponding satisfaction condition, and define a pseudo-functor LEq which assigns (among others) to a signature $\mathbf{\Sigma}$, the discrete category associated to the set of all labelled $\mathbf{\Sigma}$-equations, that enables us to obtain the many-sorted equational institution $\mathfrak{LEq}$.

After having done that we define, for the generalized terms, on the one hand, the concept of many-sorted specification and, on the other hand, that of many-sorted specification morphism, from which we obtain the corresponding category, denoted by **Spf**. Then by extending some of the notions and constructions previously developed for the category **Sig** to the category **Spf**, we obtain $\mathfrak{Spf}$, the many-sorted specification institution on **Set**. Besides, we prove that there exists a morphism from $\mathfrak{Spf}$ to $\mathfrak{Tm}$, the many-sorted term institution on **Set**, which, together with the canonical embedding of $\mathfrak{Tm}$ into $\mathfrak{Spf}$, makes of $\mathfrak{Tm}$ a retract of $\mathfrak{Spf}$.

We now define the equations over a given signature through the morphisms of the category of terms for the signature, what it means for an equation to be valid in an algebra, and the consequence operator on the many-sorted set of the equations.

Definition 3.1. Let $\mathbf{\Sigma}$ be a signature, X, Y two S-sorted sets and $\mathbf{A}$ a $\mathbf{\Sigma}$-algebra. Then a $\mathbf{\Sigma}$-*equation of type* (X,Y) is a pair $(P,Q)\colon X \longrightarrow Y$ of parallel morphisms in $\mathbf{Ter}(\mathbf{\Sigma})$ (hence $(P,Q) \in \mathrm{Hom}(Y, \mathrm{T}_{\mathbf{\Sigma}}(X))^2$), and a $\mathbf{\Sigma}$-*equation* is a $\mathbf{\Sigma}$-equation of type (X,Y) for some S-sorted sets X, Y. We will denote by $\mathrm{Eq}(\mathbf{\Sigma})$ the $(\boldsymbol{\mathcal{U}}^S)^2$-sorted set of all $\mathbf{\Sigma}$-equations. A $\mathbf{\Sigma}$-equation $(P,Q)\colon X \longrightarrow Y$ is *valid* in $\mathbf{A}$, denoted by $\mathbf{A} \models^{\mathbf{\Sigma}}_{X,Y} (P,Q)$, if and only if, for every $s \in S$ and every $y \in Y_s$, we have that $\mathbf{A} \models^{\mathbf{\Sigma}}_{X,s} (P_s(y), Q_s(y))$, i.e., that $(P_s(y))^{\mathbf{A}} = (Q_s(y))^{\mathbf{A}}$. We extend this satisfaction relation between $\mathbf{\Sigma}$-algebras $\mathbf{A}$ and $\mathbf{\Sigma}$-equations $(P,Q)\colon X \longrightarrow Y$ to $\mathbf{\Sigma}$-algebras $\mathbf{A}$ and families $\mathcal{E} \subseteq \mathrm{Eq}(\mathbf{\Sigma})$ by agreeing that $\mathbf{A} \models^{\mathbf{\Sigma}} \mathcal{E}$ if and only if, for every $X, Y \in \boldsymbol{\mathcal{U}}^S$ and $(P,Q) \in \mathcal{E}_{X,Y}$, we have that $\mathbf{A} \models^{\mathbf{\Sigma}}_{X,Y} (P,Q)$. We will denote by $\mathrm{Cn}_{\mathbf{\Sigma}}$ the endomapping of $\mathrm{Sub}(\mathrm{Eq}(\mathbf{\Sigma}))$, the set of all sub-$(\boldsymbol{\mathcal{U}}^S)^2$-sorted sets of

$\mathrm{Eq}(\mathbf{\Sigma})$, which sends $\mathcal{E} \subseteq \mathrm{Eq}(\mathbf{\Sigma})$ to $\mathrm{Cn}_{\mathbf{\Sigma}}(\mathcal{E})$, where, for every $X, Y \in \boldsymbol{\mathcal{U}}^S$ and $(P,Q) \in \mathrm{Eq}(\mathbf{\Sigma})_{X,Y}$, $(P,Q) \in \mathrm{Cn}_{\mathbf{\Sigma}}(\mathcal{E})_{X,Y}$ if and only if, for every $\mathbf{\Sigma}$-algebra $\mathbf{A}$, if $\mathbf{A} \models^{\mathbf{\Sigma}} \mathcal{E}$, then $\mathbf{A} \models^{\mathbf{\Sigma}}_{X,Y} (P,Q)$. We call $\mathrm{Cn}_{\mathbf{\Sigma}}(\mathcal{E})$ the $(\boldsymbol{\mathcal{U}}^S)^2$-sorted set of the *semantical consequences* of $\mathcal{E}$.

If we keep in mind that for a term $P\colon X \longrightarrow Y$ for $\mathbf{\Sigma}$ of type (X,Y), $P^{\mathbf{A}}$, the term operation on $\mathbf{A}$ determined by P, is the mapping from A_X to A_Y which assigns to an S-sorted mapping $f\colon X \longrightarrow A$ precisely $f^\sharp \circ P\colon Y \longrightarrow A$, then we obtain the following convenient characterization of the relation $\mathbf{A} \models^{\mathbf{\Sigma}}_{X,Y} (P,Q)$:

$$\mathbf{A} \models^{\mathbf{\Sigma}}_{X,Y} (P,Q) \text{ iff } P^{\mathbf{A}} = Q^{\mathbf{A}}.$$

Besides, by the Completeness Theorem in [5], for $\mathcal{E} \subseteq \mathrm{Eq}(\mathbf{\Sigma})$, we have that $\mathrm{Cn}_{\mathbf{\Sigma}}(\mathcal{E})$ is precisely $\mathrm{Cg}^{\Pi}_{\mathbf{Ter}(\mathbf{\Sigma})}(\mathcal{E})$, i.e., the smallest Π-compatible congruence on $\mathbf{Ter}(\mathbf{\Sigma})$ that contains $\mathcal{E}$, where the superscript Π in the operator $\mathrm{Cg}^{\Pi}_{\mathbf{Ter}(\mathbf{\Sigma})}$ abbreviates "product". Therefore the operator $\mathrm{Cn}_{\mathbf{\Sigma}}$ on the $(\boldsymbol{\mathcal{U}}^S)^2$-sorted set $\mathrm{Eq}(\mathbf{\Sigma})$ is a closure operator.

Remark. It is true that, for a signature $\mathbf{\Sigma}$, in order to equationally characterize the varieties (resp., the finitary varieties) of $\mathbf{\Sigma}$-algebras it is enough to consider the S-finite (resp., the finite) subsets of a fixed S-sorted set V^S with a countable infinity of variables in each coordinate. However, the generalized terms and equations proposed in this article, besides containing as particular cases the ordinary terms and equations, respectively, have proved their worth, e.g., in the proof of the Completeness Theorem for monads in categories of sorted sets in [5], and can also be used to attain a truly category-theoretic understanding of the subject matter (through the theory of monads).

Remark. The concept of equational deduction can be explained, from the standpoint of category theory, as a pseudo-functor. Actually, it is enough to define: (1) the category $\mathbf{MClSp}$, of many-sorted closure spaces, (2), for a Grothendieck universe $\boldsymbol{\mathcal{V}}$ such that $\boldsymbol{\mathcal{U}} \in \boldsymbol{\mathcal{V}}$, the 2-category $\mathbf{Mnd}_{\boldsymbol{\mathcal{V}},\mathrm{alg}}$ of monads (i.e., pairs $(\mathbf{C}, \mathbb{T})$ such that $\mathbf{C}$, the underlying category of the monad, is in $\boldsymbol{\mathcal{V}}$ and $\mathbb{T}$ is a monad in $\mathbf{C}$), algebraic morphisms between monads (which are adjoint squares satisfying a compatibility condition), and transformations between algebraic morphisms (which are a special type of adjoint square satisfying an additional condition), into which the category $\mathbf{MClSp}$ is naturally embedded, and (3) to prove the existence of a pseudo-functor Cn from $\mathbf{Sig}$ to $\mathbf{Mnd}_{\boldsymbol{\mathcal{V}},\mathrm{alg}}$ that has as components, essentially, the consequence operators $\mathrm{Cn}_{\mathbf{\Sigma}}$ for the different signatures $\mathbf{\Sigma}$.

By recalling that every signature morphism $\mathbf{d}$ from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$ determines a functor $\mathbf{d}_\diamond$ from $\mathbf{Ter}(\mathbf{\Sigma})$ to $\mathbf{Ter}(\mathbf{\Lambda})$, and taking into account the above definition of the equations for a signature, we next formalize the procedure of translation, by means of a signature morphism, of equations for a signature into equations for another signature in the following definition.

Definition 3.2. Let $\mathbf{d}$ be a signature morphism from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$. Then we have that $\mathbf{d}$ induces a many-sorted mapping $((\coprod_\varphi)^2, \mathbf{d}_\diamond^2)$ from $((\boldsymbol{\mathcal{U}}^S)^2, \mathrm{Eq}(\mathbf{\Sigma}))$ to $((\boldsymbol{\mathcal{U}}^T)^2, \mathrm{Eq}(\mathbf{\Lambda}))$, the so-called *translation of equations for* $\mathbf{\Sigma}$ *into equations for* $\mathbf{\Lambda}$ *relative to* $\mathbf{d}$, where $(\coprod_\varphi)^2$ is the mapping from $(\boldsymbol{\mathcal{U}}^S)^2$ to $(\boldsymbol{\mathcal{U}}^T)^2$ which sends a pair of S-sorted sets (X,Y) to the pair $(\coprod_\varphi X, \coprod_\varphi Y)$ of T-sorted sets, and $\mathbf{d}_\diamond^2$ the $(\boldsymbol{\mathcal{U}}^S)^2$-sorted mapping which to a $\mathbf{\Sigma}$-equation (P,Q) of type (X,Y) assigns the $\mathbf{\Lambda}$-equation $(\mathbf{d}_\diamond(P), \mathbf{d}_\diamond(Q))$ of type $(\coprod_\varphi X, \coprod_\varphi Y)$.

Once defined the translation of equations, we prove in the following lemma the invariance of the relation of satisfaction under signature change, also known, for those following the terminology coined by Goguen and Burstall in [12], p. 229, as the *satisfaction condition*.

Lemma 3.3. *Let* $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ *be a signature morphism,* (P,Q) *a* $\mathbf{\Sigma}$*-equation of type* (X,Y) *and* $\mathbf{A}$ *a* $\mathbf{\Lambda}$*-algebra. Then we have that*

$$\mathbf{d}^*(\mathbf{A}) \models^{\mathbf{\Sigma}}_{X,Y} (P,Q) \textit{ iff } \mathbf{A} \models^{\mathbf{\Lambda}}_{\coprod_\varphi X, \coprod_\varphi Y} (\mathbf{d}_\diamond(P), \mathbf{d}_\diamond(Q)).$$

Proof. We know that $\mathbf{d}^*(\mathbf{A}) \models^{\mathbf{\Sigma}}_{X,Y} (P,Q)$ and $P^{\mathbf{d}^*(\mathbf{A})} = Q^{\mathbf{d}^*(\mathbf{A})}$ are equivalent. In addition, by Proposition 2.17, it is true that $P^{\mathbf{d}^*(\mathbf{A})} = Q^{\mathbf{d}^*(\mathbf{A})}$ is equivalent to $\mathbf{d}_\diamond(P)^{\mathbf{A}} = \mathbf{d}_\diamond(Q)^{\mathbf{A}}$. Therefore we can assert that

$$\mathbf{d}^*(\mathbf{A}) \models^{\mathbf{\Sigma}}_{X,Y} (P,Q) \text{ iff } \mathbf{A} \models^{\mathbf{\Lambda}}_{\coprod_\varphi X, \coprod_\varphi Y} (\mathbf{d}_\diamond(P), \mathbf{d}_\diamond(Q)).$$

■

Related to the quasi-triviality of the (short and conceptual) proof of Lemma 3.3 (as a consequence, essentially, of the fact that it is, ultimately, rooted in Proposition 2.17), perhaps it would be convenient to recall that Goguen and Burstall, in [12], p. 228, have omitted the corresponding proof because they qualify it as being not entirely trivial.

To construct the many-sorted equational institution we now define a pseudo-functor LEq on the category of signatures. In order to do so we need to assume, besides the Grothendieck universe $\boldsymbol{\mathcal{U}}$, the existence of another one $\boldsymbol{\mathcal{V}}$ such that $\boldsymbol{\mathcal{U}} \in \boldsymbol{\mathcal{V}}$. The new Grothendieck universe $\boldsymbol{\mathcal{V}}$ will be used to construct the appropriate target 2-categories. Therefore, to exclude any misunderstanding, we agree to denote those categories $\mathbf{C}$ properly depending on $\boldsymbol{\mathcal{V}}$ by $\mathbf{C}_{\boldsymbol{\mathcal{V}}}$. However, since the additional assumption of a universe $\boldsymbol{\mathcal{V}}$ such that $\boldsymbol{\mathcal{U}} \in \boldsymbol{\mathcal{V}}$, will be used exclusively in this section, we do not label those categories depending on $\boldsymbol{\mathcal{U}}$ with the subscript $\boldsymbol{\mathcal{U}}$, such as has been done until now.

Definition 3.4. We denote by LEq the pseudo-functor from $\mathbf{Sig}$ to $\mathbf{Cat}_{\boldsymbol{\mathcal{V}}}$ given by the following data

1. The object mapping of LEq is that which sends a signature $\boldsymbol{\Sigma}$ to the discrete category $\mathbf{LEq}(\boldsymbol{\Sigma})$ associated to

$$\textstyle\bigcup_{X,Y \in \boldsymbol{\mathcal{U}}} (\mathrm{Hom}(Y, \mathrm{T}_{\boldsymbol{\Sigma}}(X))^2 \times \{(X,Y)\}),$$

   the set of all *labelled $\boldsymbol{\Sigma}$-equations*, i.e., the set of all pairs $((P,Q),(X,Y))$ with (P,Q) a $\boldsymbol{\Sigma}$-equation of type (X,Y), for some $X, Y \in \boldsymbol{\mathcal{U}}$.
2. The morphism mapping of LEq is that which sends a signature morphism $\mathbf{d}$ from $\boldsymbol{\Sigma}$ to $\boldsymbol{\Lambda}$ to the functor $\mathrm{LEq}(\mathbf{d})$ from $\mathbf{LEq}(\boldsymbol{\Sigma})$ to $\mathbf{LEq}(\boldsymbol{\Lambda})$ which assigns to the labelled equation $((P,Q),(X,Y))$ in $\mathbf{LEq}(\boldsymbol{\Sigma})$ the labelled equation $\mathrm{LEq}(\mathbf{d})((P,Q),(X,Y)) = ((\mathbf{d}_\diamond(P), \mathbf{d}_\diamond(Q)), (\coprod_\varphi X, \coprod_\varphi Y))$ in $\mathbf{LEq}(\boldsymbol{\Lambda})$.

Corollary 3.5. *The quadruple* $\mathfrak{LEq} = (\mathbf{Sig}, \mathrm{Alg}, \mathrm{LEq}, (\models, \theta))$ *is an institution on* $\mathbf{2}$*, the so-called* many-sorted equational institution*, or, to abbreviate, the* equational institution.

Following this we proceed to define, on the one hand, the concept of many-sorted specification and, on the other hand, that of many-sorted specification morphism.

Definition 3.6. A *many-sorted specification* is a pair $(\boldsymbol{\Sigma}, \mathcal{E})$, where $\boldsymbol{\Sigma}$ is a signature while $\mathcal{E} \subseteq \mathrm{Eq}(\boldsymbol{\Sigma})$. A *many-sorted specification morphism* from $(\boldsymbol{\Sigma}, \mathcal{E})$ to $(\boldsymbol{\Lambda}, \mathcal{H})$ is a signature morphism $\mathbf{d}\colon \boldsymbol{\Sigma} \longrightarrow \boldsymbol{\Lambda}$ such that $\mathbf{d}^2_\diamond[\mathcal{E}] \subseteq \mathrm{Cn}_{\boldsymbol{\Lambda}}(\mathcal{H})$. Henceforth, to shorten terminology, we will say *specification* and *specification morphism* instead of *many-sorted specification* and *many-sorted specification morphism*, respectively. Besides, if in a specification $(\boldsymbol{\Sigma}, \mathcal{E})$ the set $\mathcal{E}$ of equations is closed, i.e., $\mathrm{Cn}_{\boldsymbol{\Sigma}}(\mathcal{E}) = \mathcal{E}$, then we call $(\boldsymbol{\Sigma}, \mathcal{E})$ a *theory.* To abbreviate, we write, sometimes, $\overline{\mathcal{E}}$ instead of $\mathrm{Cn}_{\boldsymbol{\Sigma}}(\mathcal{E})$.

Proposition 3.7. *The specifications and the specification morphisms determine a category denoted as* $\mathbf{Spf}$.

Proof. We restrict ourselves to prove that the composition of specification morphisms is a specification morphism. But before proving this let us notice that if $\mathbf{d}\colon \boldsymbol{\Sigma} \longrightarrow \boldsymbol{\Lambda}$ and $\mathbf{e}\colon \boldsymbol{\Lambda} \longrightarrow \boldsymbol{\Omega}$ are signature morphisms, (P, Q) a $\boldsymbol{\Sigma}$-equation of type (X, Y), and $\mathbf{C}$ a $\boldsymbol{\Omega}$-algebra, then $\mathbf{e}_\diamond(\mathbf{d}_\diamond(P))^{\mathbf{C}} = \mathbf{e}_\diamond(\mathbf{d}_\diamond(Q))^{\mathbf{C}}$ if and only if $(\mathbf{e}\circ\mathbf{d})_\diamond(P)^{\mathbf{C}} = (\mathbf{e}\circ\mathbf{d})_\diamond(Q)^{\mathbf{C}}$. Thus, for each family of $\boldsymbol{\Sigma}$-equations $\mathcal{E}$, we have that $\mathrm{Cn}_{\boldsymbol{\Omega}}(\mathbf{e}_\diamond^2[\mathbf{d}_\diamond^2[\mathcal{E}]]) = \mathrm{Cn}_{\boldsymbol{\Omega}}((\mathbf{e}\circ\mathbf{d})_\diamond^2[\mathcal{E}])$. Now, if $\mathbf{d}\colon (\boldsymbol{\Sigma}, \mathcal{E}) \longrightarrow (\boldsymbol{\Lambda}, \mathcal{H})$ and $\mathbf{e}\colon (\boldsymbol{\Lambda}, \mathcal{H}) \longrightarrow (\boldsymbol{\Omega}, \mathcal{F})$ are specification morphisms, then $\mathbf{e}_\diamond^2[\mathbf{d}_\diamond^2[\mathcal{E}]] \subseteq \mathbf{e}_\diamond^2[\mathrm{Cn}_{\boldsymbol{\Lambda}}(\mathcal{H})] \subseteq \mathrm{Cn}_{\boldsymbol{\Omega}}(\mathbf{e}_\diamond^2[\mathcal{H}]) \subseteq \mathrm{Cn}_{\boldsymbol{\Omega}}(\mathcal{F})$, from which the proposition follows. ■

Remark. The category $\mathbf{Th}_{\mathrm{b}}$ with objects the theories and morphisms between them the, so-called by Bénabou in [2], p. (sub) 27, *banal* morphisms (also known as *axiom-preserving* morphisms), is $\mathbf{Th}_{\mathrm{b}} = \int^{\mathbf{Sig}} \mathrm{Fix}\circ\mathrm{Cn}$, where Fix is the contravariant functor from $\mathbf{Mnd}_{\boldsymbol{\mathcal{V}},\mathrm{alg}}$ to $\mathbf{Cat}_{\boldsymbol{\mathcal{V}}}$ which sends a monad $(\mathbf{C}, \mathbb{T})$ for $\boldsymbol{\mathcal{V}}$, to the preordered set $\mathbf{Fix}(\mathbb{T}) = (\mathrm{Fix}(\mathbb{T}), \preccurlyeq)$, of the fixed points of $\mathbb{T}$, being $\mathrm{Fix}(\mathbb{T})$ the set of all $\mathbb{T}$-algebras (A, δ) such that the structural morphism δ from $\mathrm{T}(A)$ to A is an isomorphism, and $\preccurlyeq$ the preorder on $\mathrm{Fix}(\mathbb{T})$ defined by imposing that $(A, \delta) \preccurlyeq (A', \delta')$ iff there exists a $\mathbb{T}$-homomorphism from (A, δ) to (A', δ'). Therefore, informally speaking, we can say that the world of theories, $\mathbf{Th}_{\mathrm{b}}$, is the *totalization* over $\mathbf{Sig}$ of the fixed points of the consequences.

We state next some, obvious, relations between the categories $\mathbf{Sig}$ and $\mathbf{Spf}$. Every signature $\boldsymbol{\Sigma}$ determines the specification $(\boldsymbol{\Sigma}, \varnothing)$, the so-called *indiscrete specification*, from which we obtain an inclusion functor sp_{i} from $\mathbf{Sig}$ to $\mathbf{Spf}$ that is a left adjoint to the forgetful functor sig from $\mathbf{Spf}$ to $\mathbf{Sig}$ which sends an specification $(\boldsymbol{\Sigma}, \mathcal{E})$ to the underlying signature $\boldsymbol{\Sigma}$. Besides, $\mathbf{Sig}$ is a retract of $\mathbf{Spf}$, i.e., $\mathrm{sig}\circ\mathrm{sp}_{\mathrm{i}} = \mathrm{Id}_{\mathbf{Sig}}$. The functor sig also has a right adjoint $\mathrm{sp}_{\mathrm{d}}\colon \mathbf{Sig} \longrightarrow \mathbf{Spf}$ which sends a signature $\boldsymbol{\Sigma}$ to $(\boldsymbol{\Sigma}, \mathrm{Eq}(\boldsymbol{\Sigma}))$, the so-called *discrete specification*.

What we want now is to lift the contravariant functor Alg, which is defined on $\mathbf{Sig}$, to a contravariant defined on $\mathbf{Spf}$.

Proposition 3.8. *There exists a contravariant functor* $\mathrm{Alg}^{\mathrm{sp}}$ *from* $\mathbf{Spf}$ *to* $\mathbf{Cat}$. *Its object mapping assigns to each specification* $(\boldsymbol{\Sigma}, \mathcal{E})$ *the category* $\mathrm{Alg}^{\mathrm{sp}}(\boldsymbol{\Sigma}, \mathcal{E}) = \mathbf{Alg}(\boldsymbol{\Sigma}, \mathcal{E})$ *of its models, i.e., the full subcategory of* $\mathbf{Alg}(\boldsymbol{\Sigma})$ *determined by those* $\boldsymbol{\Sigma}$*-algebras which satisfy all the equations in* $\mathcal{E}$*; its morphism mapping assigns to each specification morphism* $\mathbf{d}$ *from* $(\boldsymbol{\Sigma}, \mathcal{E})$ *to* $(\boldsymbol{\Lambda}, \mathcal{H})$ *the functor* $\mathrm{Alg}^{\mathrm{sp}}(\mathbf{d}) = \mathbf{d}^*$ *from* $\mathbf{Alg}(\boldsymbol{\Lambda}, \mathcal{H})$ *to* $\mathbf{Alg}(\boldsymbol{\Sigma}, \mathcal{E})$*, obtained from the functor* $\mathbf{d}^*$ *from* $\mathbf{Alg}(\boldsymbol{\Lambda})$ *to* $\mathbf{Alg}(\boldsymbol{\Sigma})$ *by bi-restriction.*

PROOF. Let $\mathbf{B}$ be a $\mathbf{\Lambda}$-algebra such that $\mathbf{B} \models^{\mathbf{\Lambda}} \mathcal{H}$. Then $\mathbf{B} \models^{\mathbf{\Lambda}} \mathrm{Cn}_{\mathbf{\Lambda}}(\mathcal{H})$, therefore $\mathbf{B} \models^{\mathbf{\Lambda}} \mathbf{d}^2_{\diamond}[\mathcal{E}]$ hence, by Lemma 3.3, $\mathbf{d}^*(\mathbf{B}) \models^{\mathbf{\Sigma}} \mathcal{E}$. ∎

By applying the EG-construction to the contravariant functor $\mathrm{Alg}^{\mathrm{sp}}$ we obtain the category $\int^{\mathbf{Spf}} \mathrm{Alg}^{\mathrm{sp}}$, denoted by $\mathbf{Alg}^{\mathrm{sp}}$. The category $\mathbf{Alg}^{\mathrm{sp}}$ has as objects the pairs $((\mathbf{\Sigma}, \mathcal{E}), \mathbf{A})$, where $(\mathbf{\Sigma}, \mathcal{E})$ is a specification and $\mathbf{A}$ a $\mathbf{\Sigma}$-algebra which is a model of $\mathcal{E}$, and as morphisms from $((\mathbf{\Sigma}, \mathcal{E}), \mathbf{A})$ to $((\mathbf{\Lambda}, \mathcal{H}), \mathbf{B})$, the pairs $(\mathbf{d}, f)$, with $\mathbf{d}$ a specification morphism from $(\mathbf{\Sigma}, \mathcal{E})$ to $(\mathbf{\Lambda}, \mathcal{H})$ and f a $\mathbf{\Sigma}$-homomorphism from $\mathbf{A}$ to $\mathbf{d}^*(\mathbf{B})$.

REMARK. The category **Alg** is embedded into $\mathbf{Alg}^{\mathrm{sp}}$ as a retract (essentially, because **Sig** is a retract of **Spf**).

On the other hand, taking care of the Completeness Theorem in [5], every family of equations $\mathcal{E} \subseteq \mathrm{Eq}(\mathbf{\Sigma})$ determines a congruence on the category $\mathbf{Ter}(\mathbf{\Sigma})$, hence a quotient category $\mathbf{Ter}(\mathbf{\Sigma})/\overline{\mathcal{E}}$. Besides, this procedure can be completed, as stated in the following proposition, to a pseudo-functor $\mathrm{Ter}^{\mathrm{sp}}$ from **Spf** to **Cat**, and the restriction of $\mathrm{Ter}^{\mathrm{sp}}$ to **Sig** is precisely the pseudo-functor Ter.

PROPOSITION 3.9. *There exists a pseudo-functor* $\mathrm{Ter}^{\mathrm{sp}}$ *from* **Spf** *to* **Cat** *defined as follows*

1. $\mathrm{Ter}^{\mathrm{sp}}$ *sends a specification* $(\mathbf{\Sigma}, \mathcal{E})$ *in* **Spf** *to the category* $\mathrm{Ter}^{\mathrm{sp}}(\mathbf{\Sigma}, \mathcal{E}) = \mathbf{Ter}(\mathbf{\Sigma}, \mathcal{E})$*, where* $\mathbf{Ter}(\mathbf{\Sigma}, \mathcal{E})$ *is the quotient category* $\mathbf{Ter}(\mathbf{\Sigma})/\overline{\mathcal{E}}$.
2. $\mathrm{Ter}^{\mathrm{sp}}$ *sends a specification morphism* $\mathbf{d}$ *from* $(\mathbf{\Sigma}, \mathcal{E})$ *to* $(\mathbf{\Lambda}, \mathcal{H})$ *to the functor* $\mathrm{Ter}^{\mathrm{sp}}(\mathbf{d})$*, also occasionally denoted by* $\mathbf{d}_{\diamond}$*, from* $\mathbf{Ter}(\mathbf{\Sigma}, \mathcal{E}) = \mathbf{Ter}(\mathbf{\Sigma})/\overline{\mathcal{E}}$ *to* $\mathbf{Ter}(\mathbf{\Lambda}, \mathcal{H}) = \mathbf{Ter}(\mathbf{\Lambda})/\overline{\mathcal{H}}$*, which assigns to a morphism* $[P]_{\overline{\mathcal{E}}}$ *from* X *to* Y *in* $\mathbf{Ter}(\mathbf{\Sigma}, \mathcal{E})$ *the morphism* $\mathrm{Ter}^{\mathrm{sp}}(\mathbf{d})([P]_{\overline{\mathcal{E}}}) = [\mathbf{d}_{\diamond}(P)]_{\overline{\mathcal{H}}}$ *from* $\coprod_{\varphi} X$ *to* $\coprod_{\varphi} Y$ *in* $\mathbf{Ter}(\mathbf{\Lambda}, \mathcal{H})$.

PROOF. Everything follows, essentially, from the fact that the action of $\mathrm{Ter}^{\mathrm{sp}}(\mathbf{d})$ on $[P]_{\overline{\mathcal{E}}}$ is well defined since $\mathcal{E} \subseteq \mathrm{Ker}(\mathrm{Pr}_{\overline{\mathcal{H}}} \circ \mathbf{d}_{\diamond})$, where $\mathrm{Pr}_{\overline{\mathcal{H}}}$ is the projection from $\mathbf{Ter}(\mathbf{\Lambda})$ to $\mathbf{Ter}(\mathbf{\Lambda})/\overline{\mathcal{H}}$. ∎

After this we state that the family of functors $\mathrm{Tr} = (\mathrm{Tr}^{\mathbf{\Sigma}})_{\mathbf{\Sigma} \in \mathbf{Sig}}$, defined in Proposition 2.23, can be lifted to the family of functors $\mathrm{Tr}^{\mathrm{sp}} = (\mathrm{Tr}^{\mathrm{sp},(\mathbf{\Sigma},\mathcal{E})})_{(\mathbf{\Sigma},\mathcal{E}) \in \mathbf{Spf}}$.

PROPOSITION 3.10. *Let* $(\mathbf{\Sigma}, \mathcal{E})$ *be a specification. Then from the product category* $\mathbf{Alg}(\mathbf{\Sigma}, \mathcal{E}) \times \mathbf{Ter}(\mathbf{\Sigma}, \mathcal{E})$ *to the category* **Set** *there exists a functor* $\mathrm{Tr}^{\mathrm{sp},(\mathbf{\Sigma},\mathcal{E})}$*. Its object mapping assigns to each pair* $(\mathbf{A}, X)$*, formed by a* $\mathbf{\Sigma}$*-algebra* $\mathbf{A}$ *which satisfies* $\mathcal{E}$ *and an* S*-sorted set* X*, the set* $\mathrm{Tr}^{\mathrm{sp},(\mathbf{\Sigma},\mathcal{E})}(\mathbf{A}, X) =$

A_X of the S-sorted mappings from X to the underlying S-sorted set A of $\mathbf{A}$; its morphism mapping assigns to each arrow $(f,[P]_{\overline{\mathcal{E}}})$ from $(\mathbf{A},X)$ to $(\mathbf{B},Y)$ the mapping $\mathrm{Tr}^{\mathrm{sp},(\boldsymbol{\Sigma},\mathcal{E})}(f,[P]_{\overline{\mathcal{E}}}) = f_P$ from A_X to B_Y.

PROOF. Everything follows from the fact that the action of $\mathrm{Tr}^{\mathrm{sp},(\boldsymbol{\Sigma},\mathcal{E})}$ on $(f,[P]_{\overline{\mathcal{E}}})$ is well defined because from $[P]_{\overline{\mathcal{E}}} = [Q]_{\overline{\mathcal{E}}}$ it follows that, for every $\boldsymbol{\Sigma}$-algebra $\mathbf{C}$ which satisfies $\mathcal{E}$, $P^{\mathbf{C}} = Q^{\mathbf{C}}$. ∎

Next we state that the family of natural isomorphisms $\theta = (\theta^{\mathbf{d}})_{\mathbf{d}\in\mathrm{Mor}(\mathbf{Sig})}$, defined in Proposition 2.24, can be lifted to a corresponding family of natural isomorphisms $\theta^{\mathrm{sp}} = (\theta^{\mathrm{sp},\mathbf{d}})_{\mathbf{d}\in\mathrm{Mor}(\mathbf{Spf})}$.

PROPOSITION 3.11. *Let $\mathbf{d}$ be a specification morphism from $(\boldsymbol{\Sigma},\mathcal{E})$ to $(\boldsymbol{\Lambda},\mathcal{H})$. Then there exists a natural isomorphism*

$$\theta^{\mathrm{sp},\mathbf{d}} = (\theta^{\mathrm{sp},\mathbf{d}}_{\mathbf{A},X})_{(\mathbf{A},X)\in\mathbf{Alg}(\boldsymbol{\Lambda},\mathcal{H})\times\mathbf{Ter}(\boldsymbol{\Sigma},\mathcal{E})}$$

as shown in the following diagram

$$\begin{tikzcd}
\mathbf{Alg}(\boldsymbol{\Lambda},\mathcal{H})\times\mathbf{Ter}(\boldsymbol{\Sigma},\mathcal{E}) \arrow[r, "\mathbf{d}^*\times\mathrm{Id}"] \arrow[d, "\mathrm{Id}\times\mathbf{d}_\diamond"'] \arrow[dr, phantom, "\overset{\theta^{\mathrm{sp},\mathbf{d}}}{\Rightarrow}"] & \mathbf{Alg}(\boldsymbol{\Sigma},\mathcal{E})\times\mathbf{Ter}(\boldsymbol{\Sigma},\mathcal{E}) \arrow[d, "\mathrm{Tr}^{\mathrm{sp},(\boldsymbol{\Sigma},\mathcal{E})}"] \\
\mathbf{Alg}(\boldsymbol{\Lambda},\mathcal{H})\times\mathbf{Ter}(\boldsymbol{\Lambda},\mathcal{H}) \arrow[r, "\mathrm{Tr}^{\mathrm{sp},(\boldsymbol{\Lambda},\mathcal{H})}"'] & \mathbf{Set}
\end{tikzcd}$$

where, for every $(\mathbf{A},X)\in\mathbf{Alg}(\boldsymbol{\Lambda},\mathcal{H})\times\mathbf{Ter}(\boldsymbol{\Sigma},\mathcal{E})$, $\theta^{\mathrm{sp},\mathbf{d}}_{\mathbf{A},X}$ is $\theta^{\varphi}_{X,A}$, i.e., the value at (X,A) of the natural isomorphism of the adjunction $\coprod_\varphi \dashv \Delta_\varphi$.

From these two last propositions it follows immediately the following corollary.

COROLLARY 3.12. *The quadruple $\mathfrak{Spf} = (\mathbf{Spf},\mathrm{Alg}^{\mathrm{sp}},\mathrm{Ter}^{\mathrm{sp}},(\mathrm{Tr}^{\mathrm{sp}},\theta^{\mathrm{sp}}))$ is an institution on the category $\mathbf{Set}$, the so-called* many-sorted specification institution, *or, to abbreviate, the* specification institution.

On the other hand, from the contravariant functor $\mathrm{Alg}^{\mathrm{sp}}$, from $\mathbf{Spf}$ to $\mathbf{Cat}$, to the contravariant functor $\mathrm{Alg}\circ\mathrm{sig}^{\mathrm{op}}$, between the same categories, there exists a natural transformation, In, which sends a specification $(\boldsymbol{\Sigma},\mathcal{E})$ to the full embedding $\mathrm{In}_{(\boldsymbol{\Sigma},\mathcal{E})}$ of $\mathbf{Alg}(\boldsymbol{\Sigma},\mathcal{E})$ into $\mathbf{Alg}(\boldsymbol{\Sigma})$. Besides, from the pseudo-functor $\mathrm{Ter}\circ\mathrm{sig}$, from $\mathbf{Spf}$ to $\mathbf{Cat}$, to the pseudo-functor $\mathrm{Ter}^{\mathrm{sp}}$, between the same categories, there exists a (strict) pseudo-natural transformation, Pr, given by the following data

1. For each specification $(\mathbf{\Sigma}, \mathcal{E})$, the projection functor $\mathrm{Pr}_{\overline{\mathcal{E}}}$ from $\mathbf{Ter}(\mathbf{\Sigma})$ to the quotient category $\mathbf{Ter}(\mathbf{\Sigma})/\overline{\mathcal{E}}$.
2. For each specification morphism $\mathbf{d}$ from $(\mathbf{\Sigma}, \mathcal{E})$ to $(\mathbf{\Lambda}, \mathcal{H})$, the identity natural transformation, denoted in this case by $\mathrm{Pr}_{\mathbf{d}}$, from the functor $\mathrm{Pr}_{\overline{\mathcal{H}}} \circ (\mathrm{Ter} \circ \mathrm{sig})(\mathbf{d})$ to the functor $\mathrm{Ter}^{\mathrm{sp}}(\mathbf{d}) \circ \mathrm{Pr}_{\overline{\mathcal{E}}}$, both from $\mathbf{Ter}(\mathbf{\Sigma})$ to $\mathbf{Ter}(\mathbf{\Lambda})/\overline{\mathcal{H}}$.

Therefore, for the concept of institution morphism as stated, e.g., in [8], p. 39, we have obtained the following corollary.

COROLLARY 3.13. *The pair* $(\mathrm{sig}, (\mathrm{In}, \mathrm{Pr}))$ *is a morphism of institutions from the many-sorted specification institution* $\mathfrak{Spf}$ *to the many-sorted term institution* $\mathfrak{Tm}$.

REMARK. Since, obviously, $\mathfrak{Tm}$ is embedded in $\mathfrak{Spf}$, taking into account the corollary just stated, we can assert that $\mathfrak{Tm}$ is, to be more precise, a retract of $\mathfrak{Spf}$.

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